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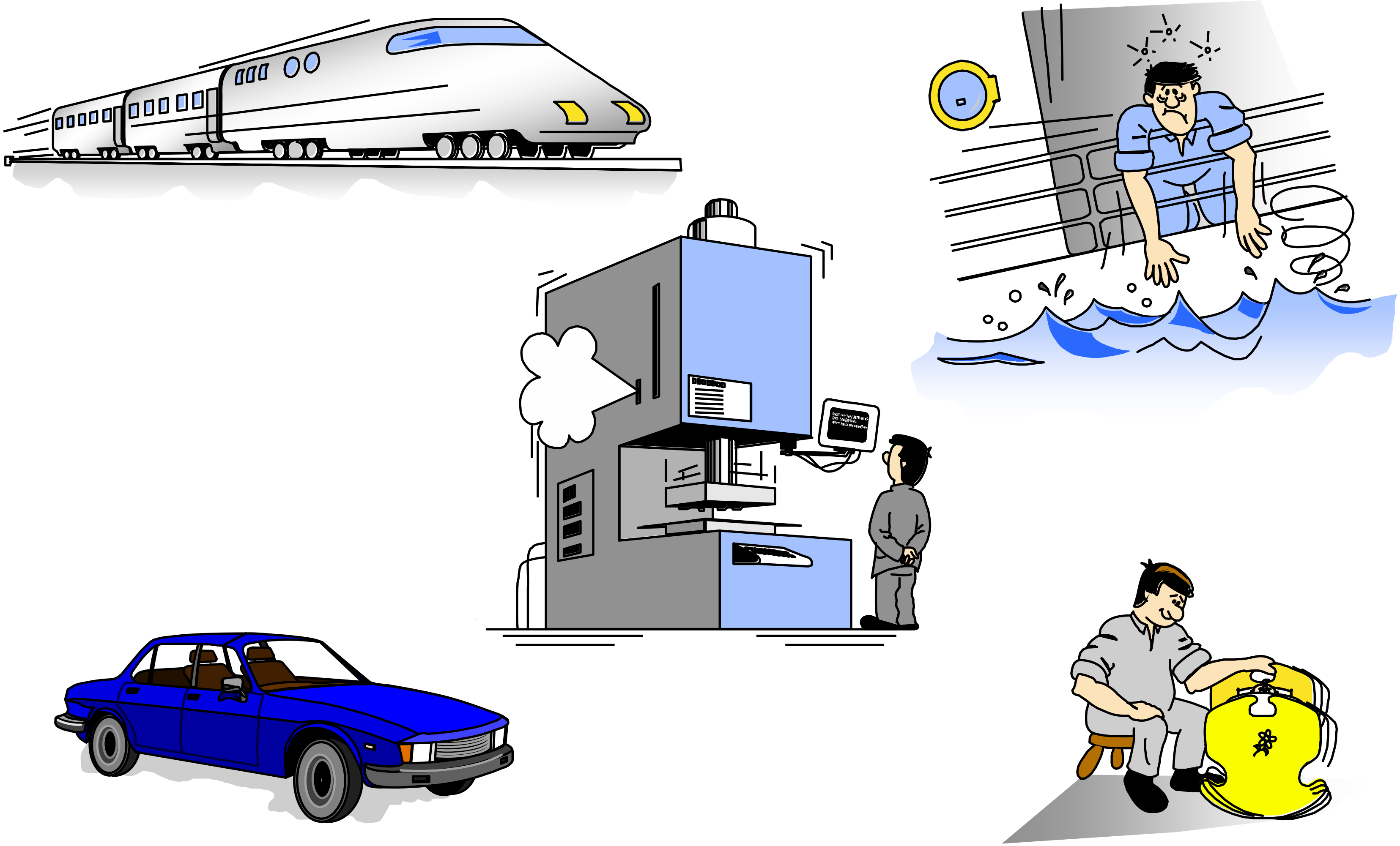
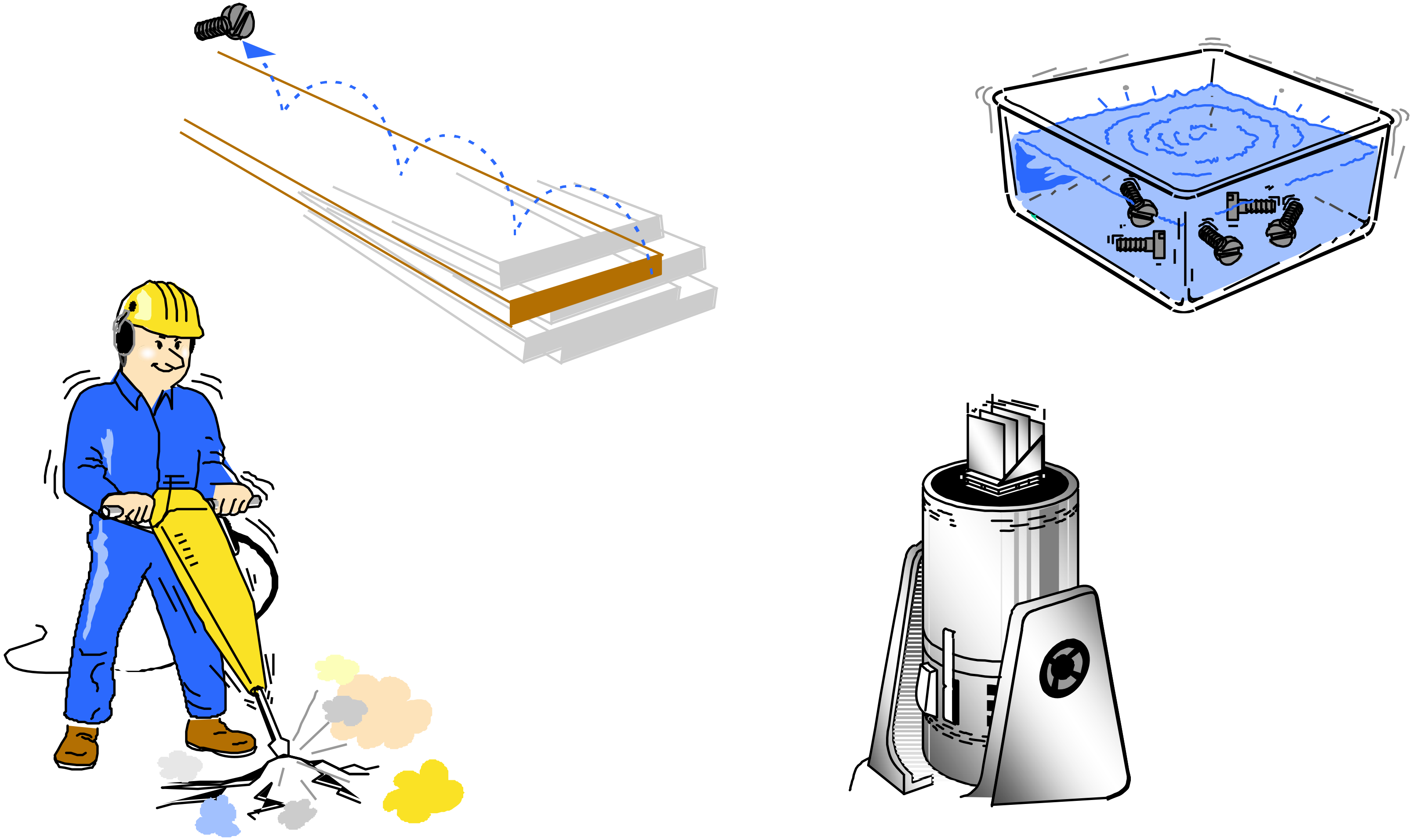
2.1 Concepts of Vibration

- Vibration In Everyday Life



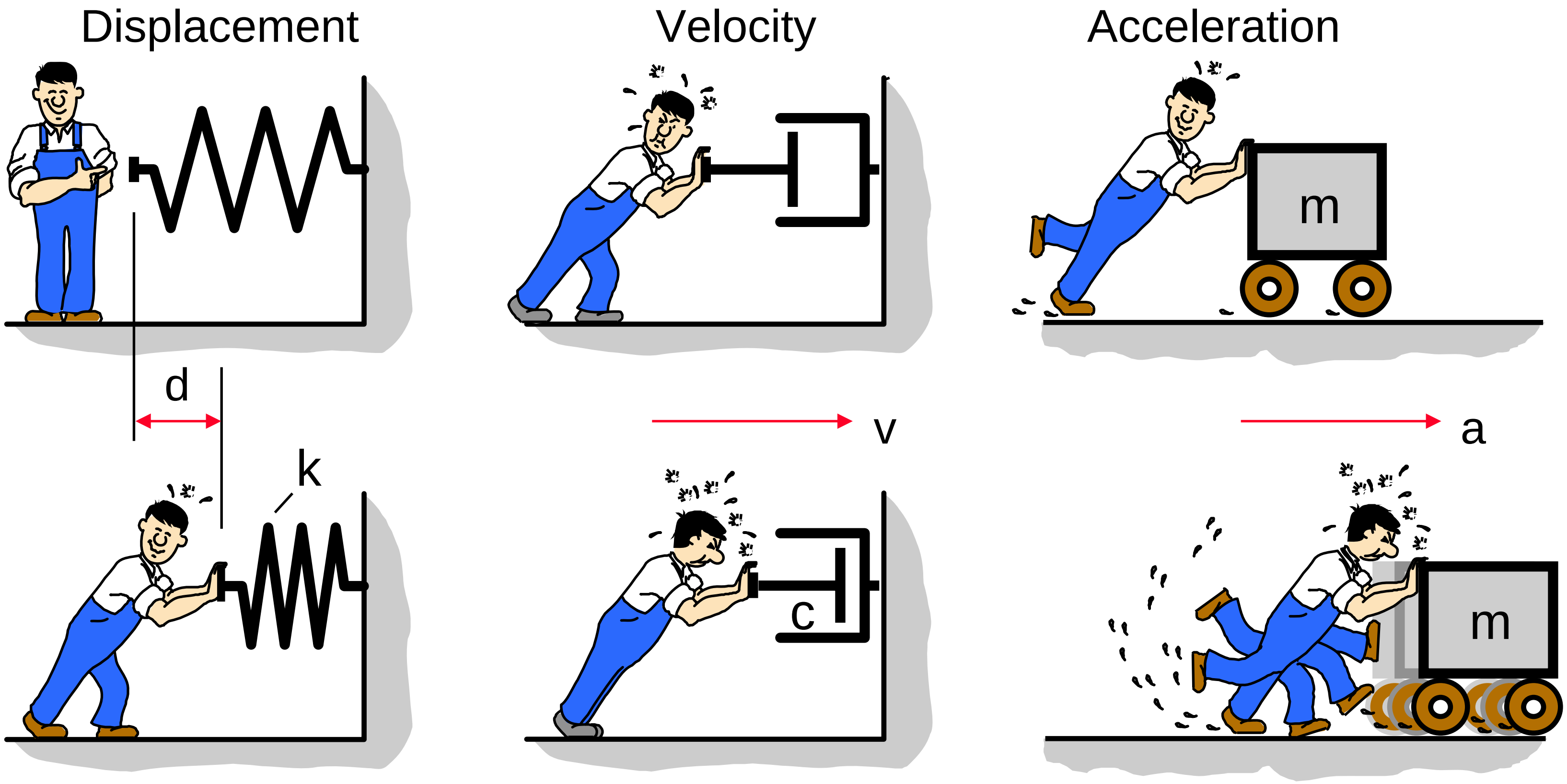
2.1 Concepts of Vibration

- Useful Vibration



2.1 Concepts of Vibration

- Mechanical Parameters and Components



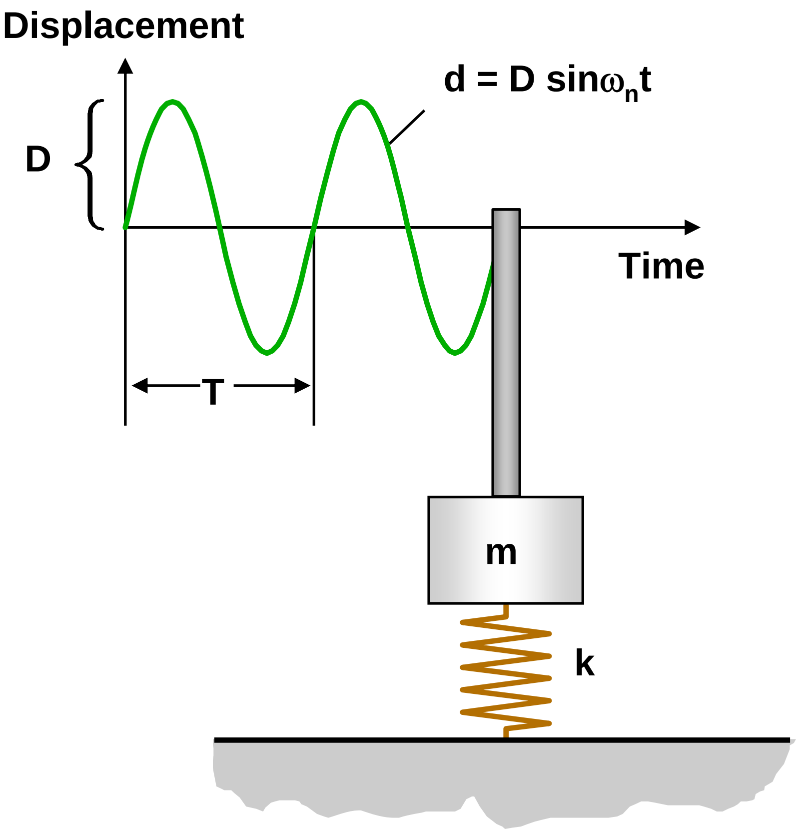
$F = k \times d$

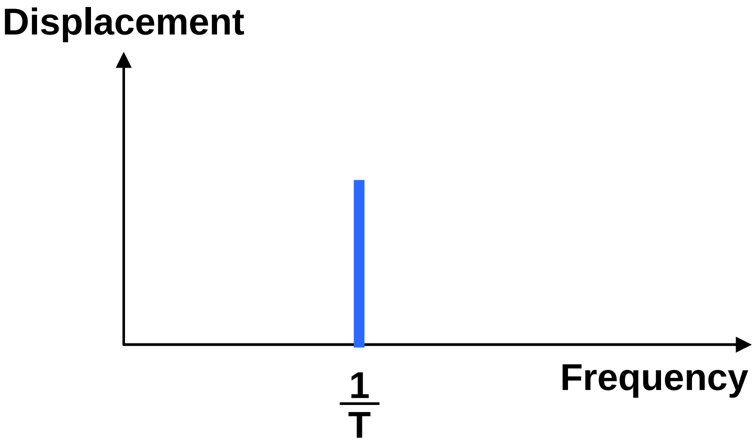
$F = c \times v$

$F = m \times a$

2.2 1 DOF system

- Simplest Form of Vibrating System





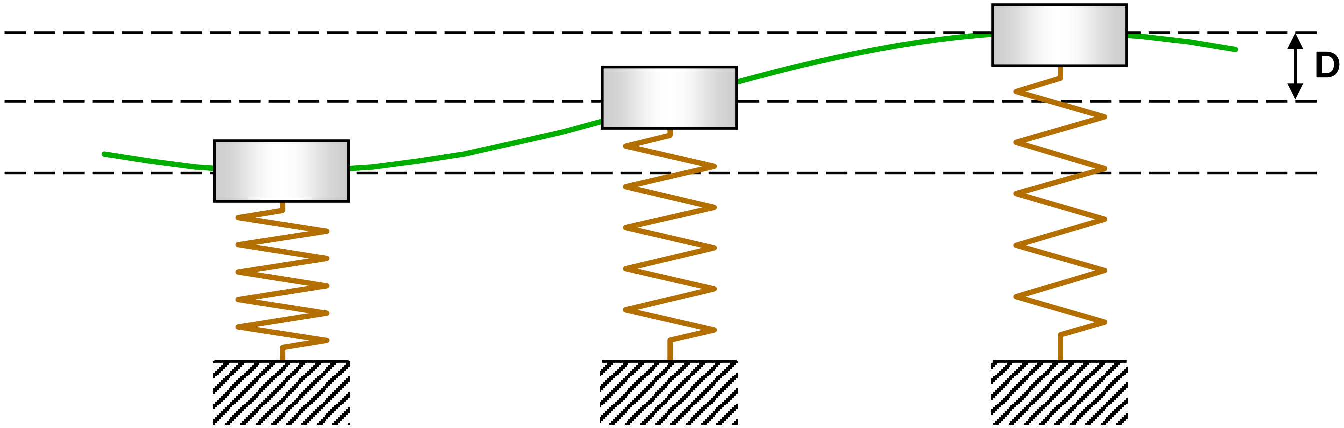
Period, T_n in [sec]

Frequency, $f_n = \frac{1}{T_n}$ **in [Hz = ^{1}/sec]**

$$\omega_n = 2\pi f_n = \sqrt{\frac{k}{m}}$$

2.2 1 DOF system

- Free Vibration



Energy transfer between Kinetic and Potential Energy
(assuming no damping)

Δ Kinetic Energy = - Δ Potential Energy

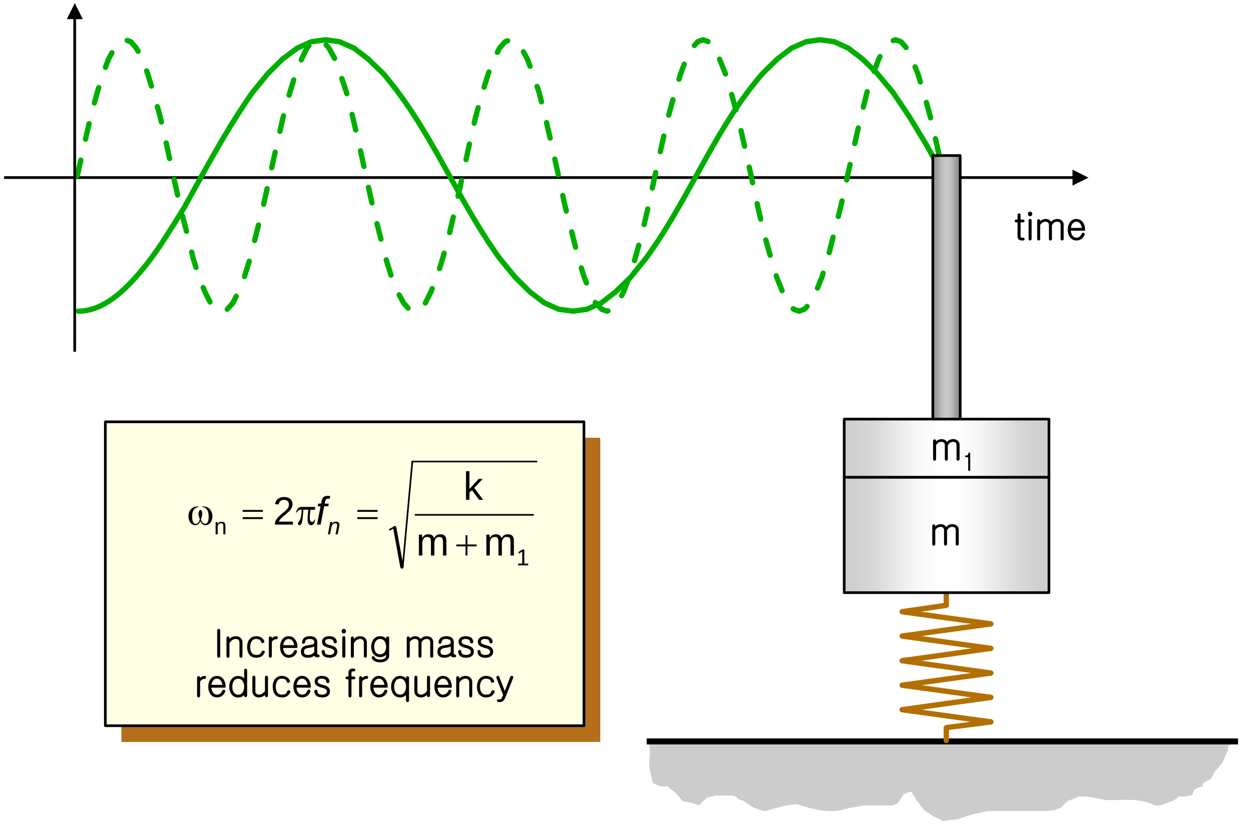
$$\frac{1}{2} m V^2 = \frac{1}{2} k D^2 \text{, and } V = (2\pi f_n)D$$

$$\frac{1}{2} m (2\pi f_n)^2 D^2 = \frac{1}{2} k D^2$$

$$\boxed{f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}}$$

2.2 1 DOF system

- Free Vibration

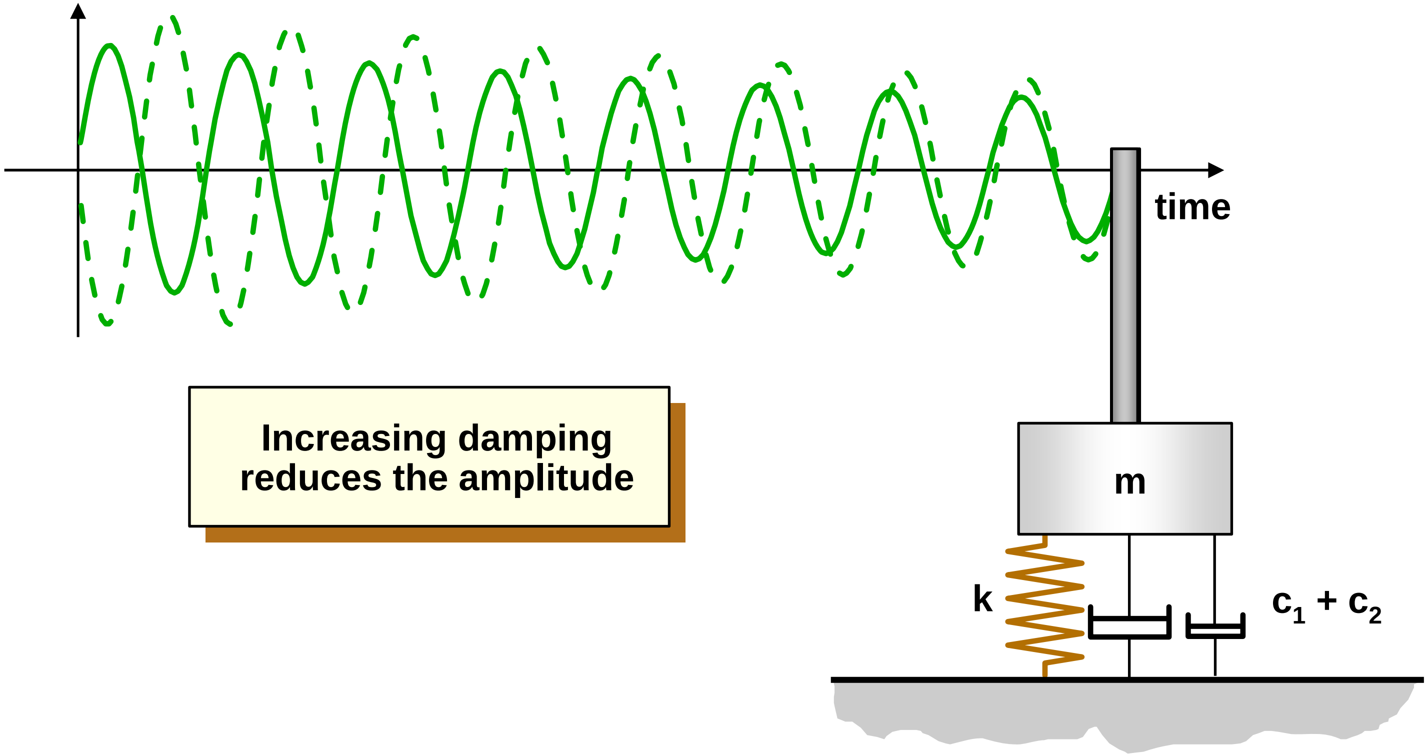


$$\omega_n = 2\pi f_n = \sqrt{\frac{k}{m+m_1}}$$

Increasing mass reduces frequency

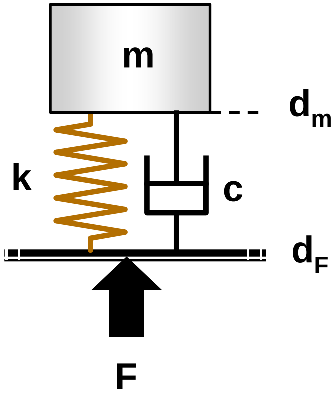
2.2 1 DOF system

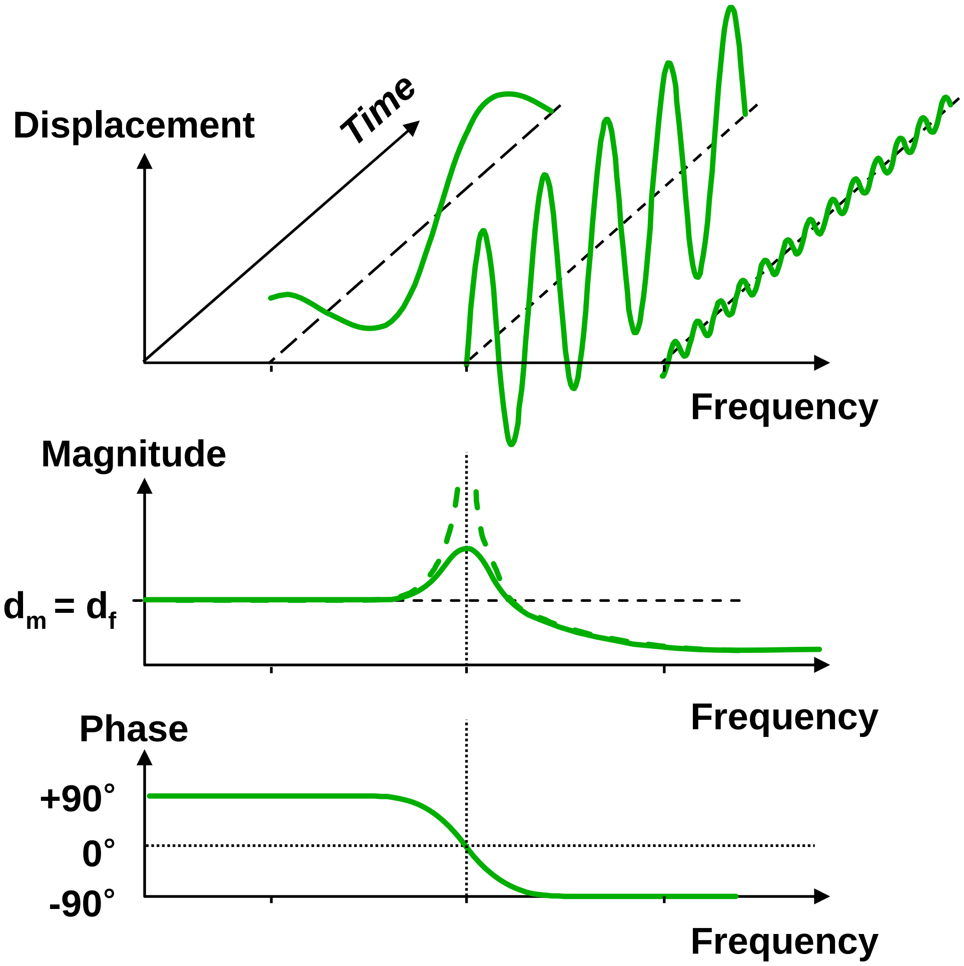
- Mass, Spring and Damper



2.2 1 DOF system

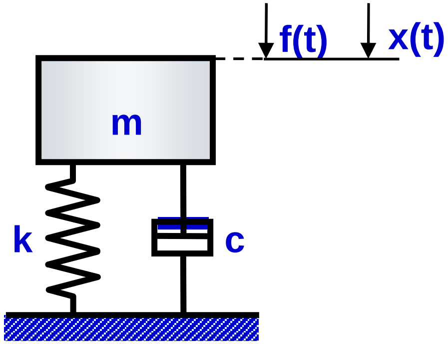
- Forced Vibration





2.2 1 DOF system

- Single Degree of Freedom Model



Newton 2nd law:

$$m\ddot{x}(t) = f(t) - kx(t) - c\dot{x}(t)$$

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t)$$

Force Balance: Inertial + Dissipative + Restoring = External

2.2 1 DOF system

- What is a Transformation?

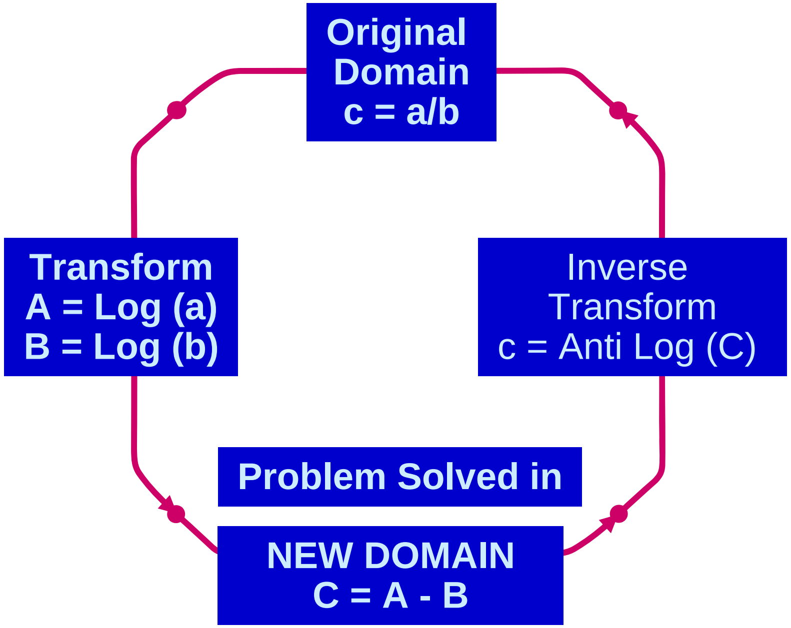
Transformation = Change of Basis

Fourier Transformation

- Why Make Transformations?
  - Simplify problem solving
  - Reduce problem size
  - Uncouple equations
  - Simplify interpretation

- Using Transformations

Example:



2.2 1 DOF system

- Examples of Transformation

Original Domain		**Transformation Domain**
c = a/b	**<Log>**	**C = A - B**
$m\ddot{x}+c\dot{x}+kx=f(t)$	**< Laplace>**	$X(s)=\dfrac{F(s)}{ms^2+cs+k}$
$x(t)=\int_{-\infty}^{\infty} h(\tau)f(t-\tau)\,d\tau$	**< Fourier>**	$X(\omega)=H(\omega)\cdot F(\omega)$
$[m]\{\ddot{x}\}+[c]\{\dot{x}\}+[k]\{x\}=\{f\}$	**< Modal>**	$\{\ddot{q}\}+[2\sigma]\{\dot{q}\}+\left[\omega_o^2\right]\{q\}=\{\Gamma(t)\}$

- Linear Transformations by Orthogonal Functions Mapping Matrix

(orthogonal)

↓

Physical Coordinates → $\{x\}=[T]\{X\}$ ← **Transform Coordinates**

2.2 1 DOF system

- Transformation

- Laplace Transform

$$G(s) = \int_0^\infty g(t)\, e^{-st}\, dt$$

s is a complex variable

$s = \sigma + j\omega$

- Fourier Transform

$$G(j\omega) = \int_{-\infty}^\infty g(t)\, e^{-j\omega t}\, dt$$

For causal f(t) i.e. $g(t) = 0,\ t < 0$

$G(j\omega) = G(s)\big|_{s=j\omega}$

- Note

  The variable σ (real part of s) should not be confused with the system constant σ called Decay Rate

2.2 1 DOF system

- Transformation

Properties of Transforms

Time $f(t)$	Fourier $F(\omega)$	Laplace $F(s)$
$a_1F_1(t) + a_2F_2(\omega)$	$a_1F_1(\omega) + a_2F_2(\omega)$	$a_1F_1(s) + a_2F_2(s)$
$\frac{df}{dt}$	$(j\omega)F(\omega)$	$s\ F(s)$ + I.C.
$\frac{d^mf}{dt^m}$	$(j\omega)^mF(\omega)$	$(s)^mF(s)$ + I.C.

I.C. = Initial Condition

2.2 1 DOF system

- Free Undamped Vibration

$$M\ddot{x}(t) + Kx(t) = 0$$

Trial solution $x(t) = Xe^{st}$

$$(Ms^2 + K)X = 0 \qquad s = \pm j\sqrt{\frac{K}{M}} = \pm j\omega_n$$

Response solution $x(t) = \dfrac{\dot{x}(0)}{\omega_n}\sin\,\omega_n t + x(0)\cos\,\omega_n t$

The system vibrates only with its natural frequency !

2.2 1 DOF system

- Free Damped Vibration

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = 0$$

Trial solution $x(t) = Xe^{st}$

$$(Ms^2 + Cs + K)X = 0 \qquad s = -\frac{C}{2M} \pm \frac{1}{2M}\sqrt{C^2 - 4MK}$$

Critical Damping $C_c = \sqrt{2MK}$

Damping Ratio $\varsigma = \dfrac{C}{C_c} = \dfrac{C}{\sqrt{2MK}}$

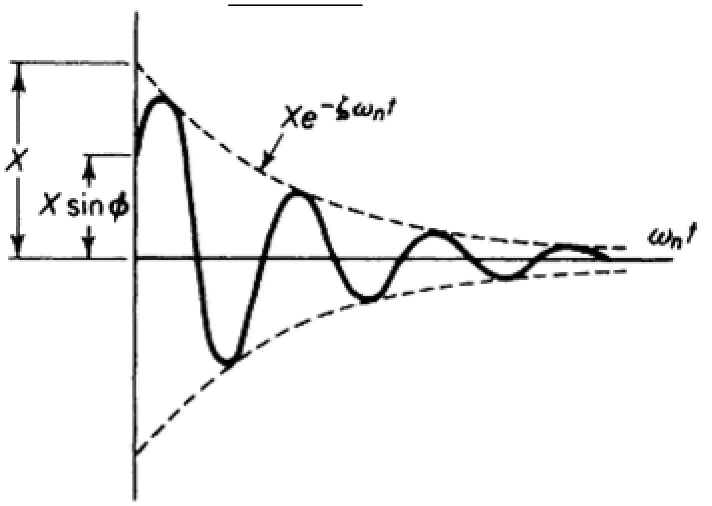
2.2 1 DOF system

Non-dimensionalized equation

$$\ddot{x}(t) + 2\varsigma\omega_n \dot{x}(t) + \omega_n^2 x(t) = 0$$

$$s = -\varsigma\omega_n \pm i\sqrt{1-\varsigma^2}\,\omega_n$$

Response solution $$x(t) = Xe^{-\varsigma\omega_n t}\sin(\omega_d t + \phi)$$



2.2 1 DOF system

- Harmonic Excited Vibration

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = F\sin\omega t \quad or \quad F\cos\omega t$$

Trial solution $x(t) = X\sin(\omega t - \phi) \quad or \quad X\cos(\omega t - \phi)$

Vectroial method

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = Fe^{i\omega t}$$

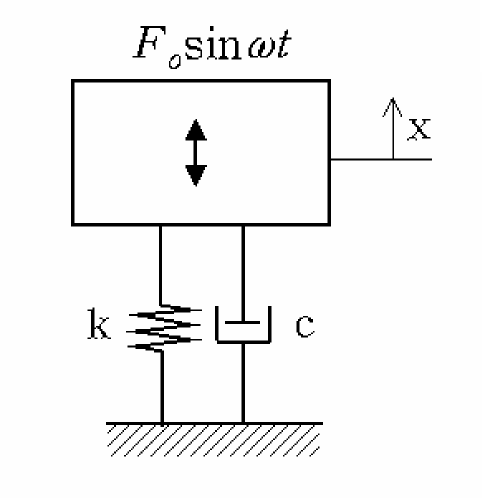
$$x(t) = \overline{X}e^{i\omega t}$$

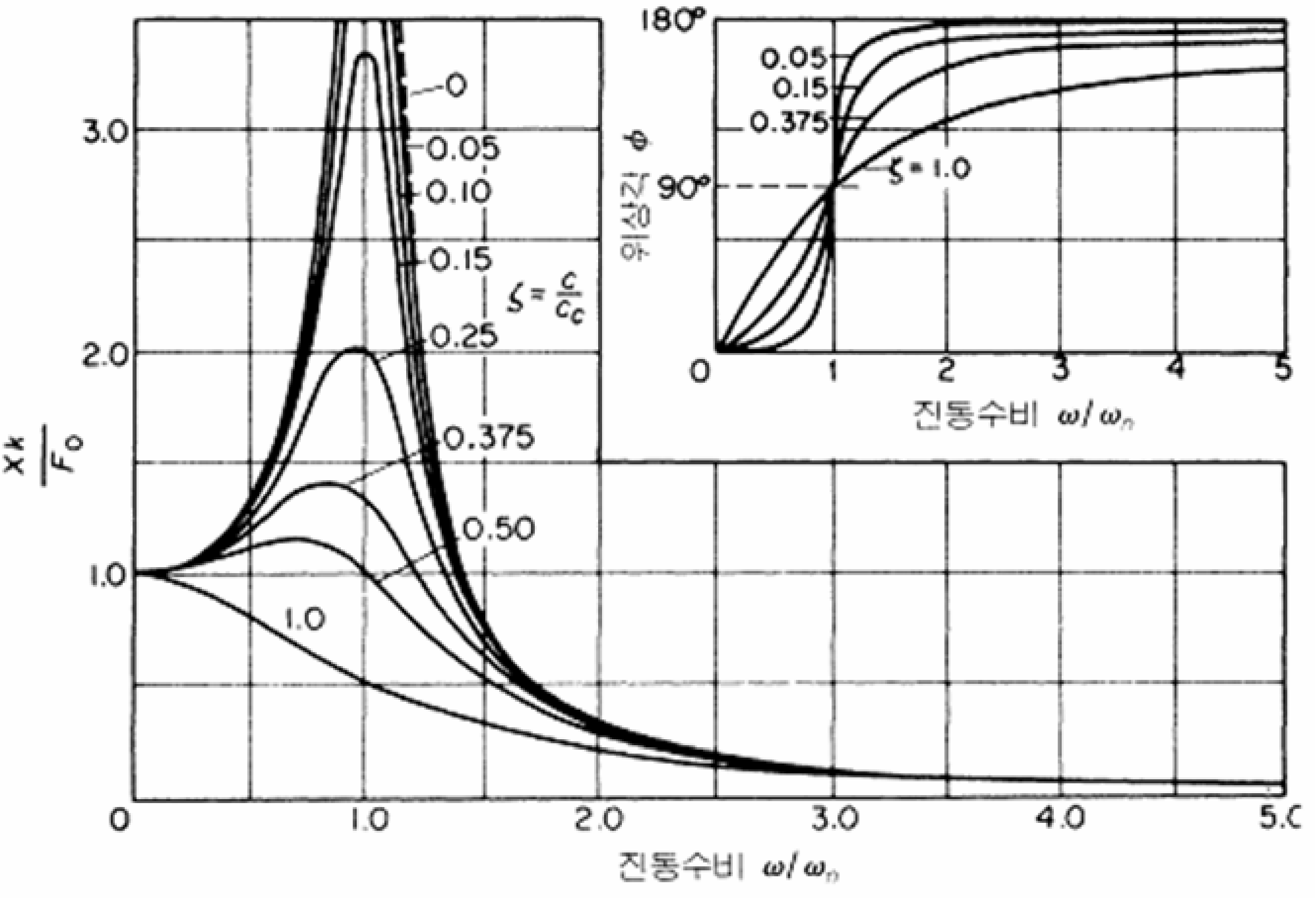
Respose solution $\overline{X} = \dfrac{F}{K - M\omega^2 + i\omega C}$

2.2 1 DOF system

- Directly Excited by force

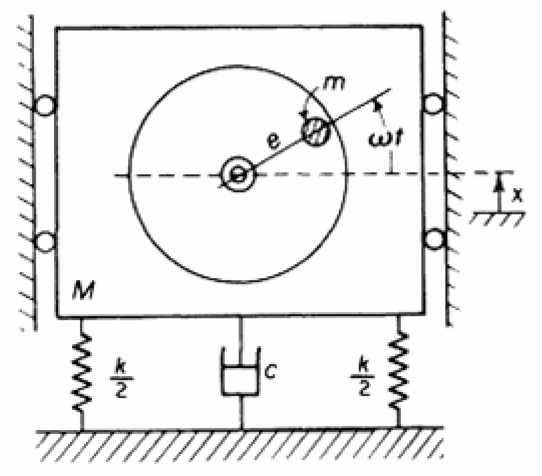
$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = F \sin \omega t$$



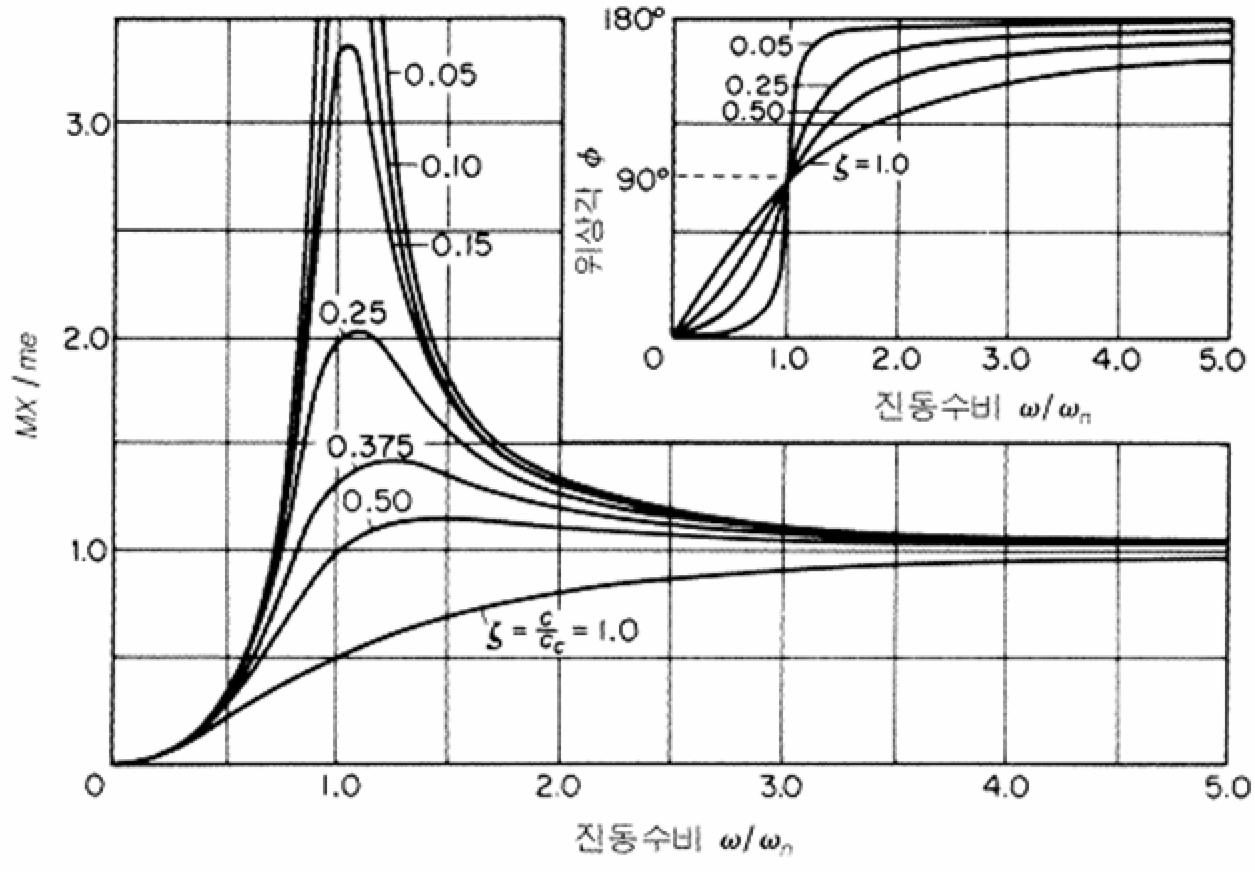


2.2 1 DOF system

- Excited by unbalance mass



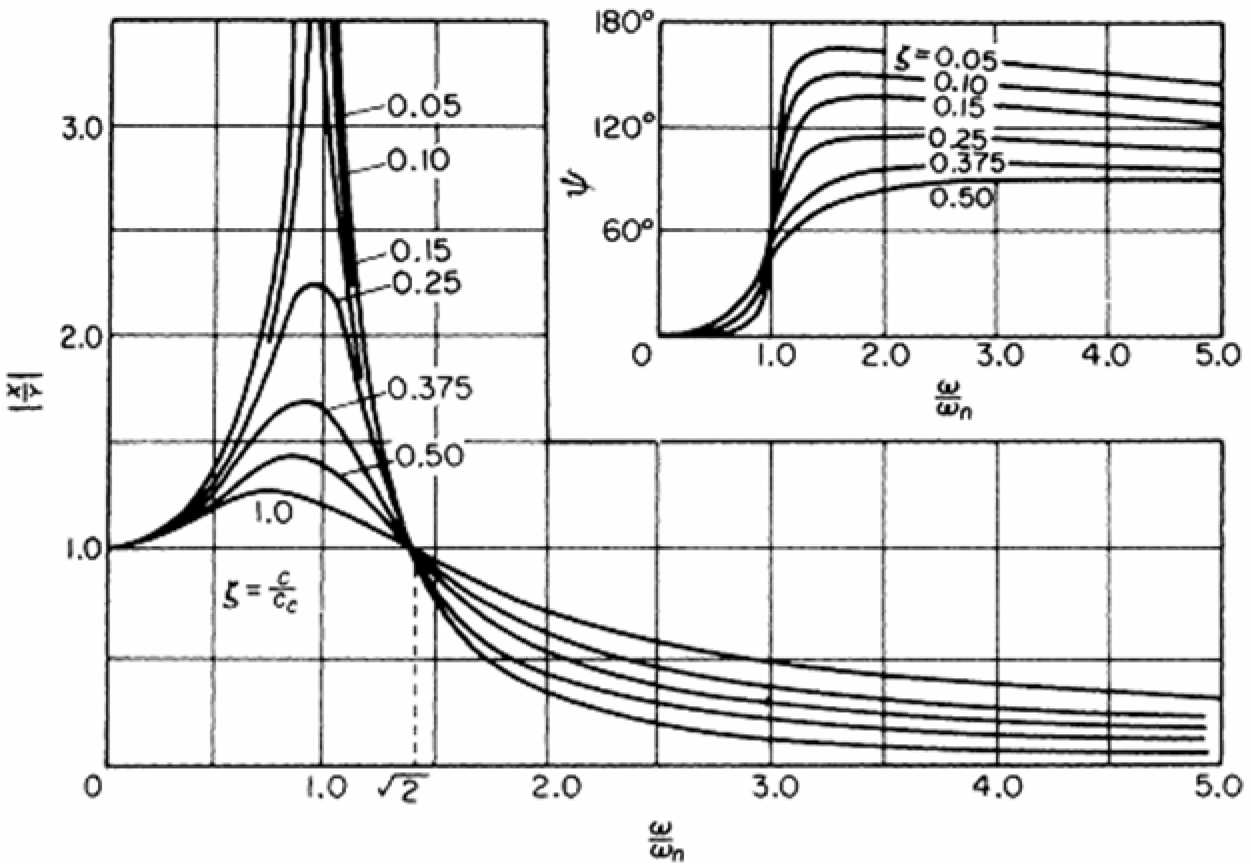
$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = me\omega^2 \sin \omega t$$



2.2 1 DOF system

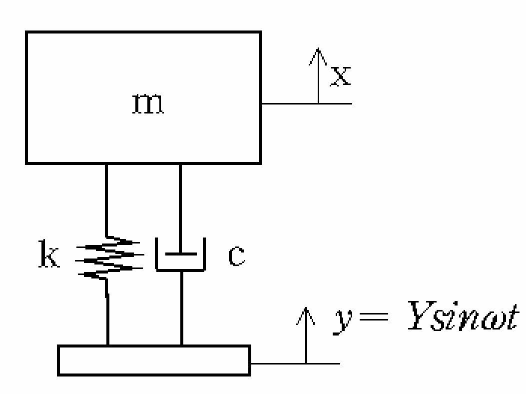
- Excited by base vibration

$$\left|\frac{X}{Y}\right| = \sqrt{\frac{k^2 + (\omega c)^2}{(k - m\omega^2)^2 + (\omega c)^2}}$$



2.2 1 DOF system

- Excited by base vibration



$$m\ddot{x} = -k(x-y) - c(\dot{x} - \dot{y})$$

Let $z = x - y$

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} = m\omega^2 Y\sin\omega t$$

$$\therefore \quad z = Z\sin(\omega t - \phi)$$

$$z = \frac{m\omega^2 Y}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}, \quad \tan\phi = \frac{c\omega}{k - m\omega^2}$$

$$x = z + y$$

$$y = Y e^{i\omega t}$$

$$z = Z e^{i(\omega t - \phi)} = (Z e^{-i\phi}) e^{i\omega t}$$

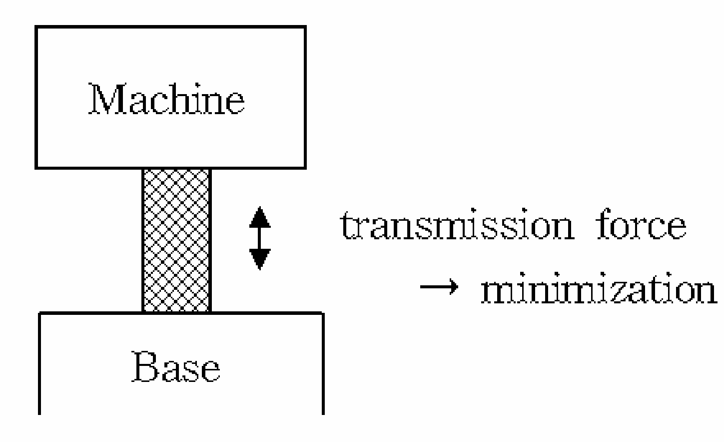
$$x = X e^{i(\omega t - \phi)} = (X e^{-i\phi}) e^{i\omega t}$$

$$Z e^{-i\phi} = \frac{m\omega^2 Y}{k - m\omega^2 + i\omega c}$$

$$x = (Z e^{-i\phi} + Y) e^{i\omega t} = \left(\frac{k + i\omega c}{k - m\omega^2 + i\omega c}\right) Y e^{i\omega t}$$

2.2 1 DOF system

- Transmission force



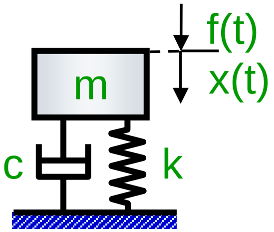
$$|f_T| = F_T = \sqrt{(kX)^2 + (c\omega X)^2} = kX\sqrt{1 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2}$$

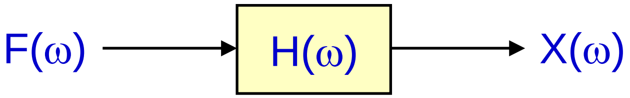
$$= F_o\sqrt{\frac{1+(2\zeta\omega/\omega_n)^2}{[1-(\omega/\omega_n)^2]^2+[2\zeta\omega/\omega_n]^2}}$$

$$\frac{F_T}{F_o} = \sqrt{\frac{1+(2\zeta\omega/\omega_n)^2}{[1-(\omega/\omega_n)^2]^2+[2\zeta\omega/\omega_n]^2}}$$: Tansmissibility

2.2 1 DOF system

- SDOF Models — time and frequency domain

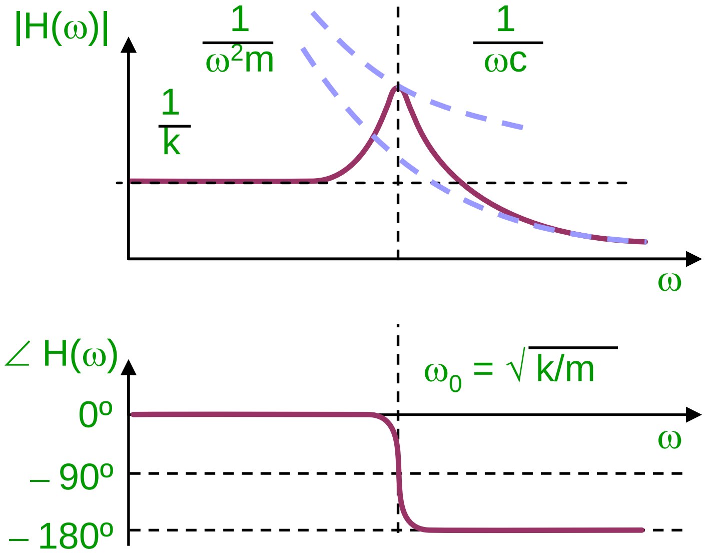




$$f(t) = m\ddot{x}(t) + c\dot{x}(t) + kx(t)$$

$$F(\omega) = -\omega^2 mX(\omega) + j\omega\, cX(\omega) + kX(\omega)$$

$$H(\omega) = \frac{X(\omega)}{F(\omega)} = \frac{1}{-\omega^2 m + j\omega c + k}$$



2.2 1 DOF system

- Laplace Domain Model — Forced Vibration

$$\boxed{[ms^2 + cs + k]\, X(S) = F(s)}$$

- The Transfer Function

$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k} = \frac{1/m}{(s-r)(s-r^*)}$$

$r, r^* = -\sigma \pm j\omega_d$ are the poles p for H(s) $\quad i.e.\ \ H(s) = \dfrac{1/m}{(s-p)\,(s-p^*)}$

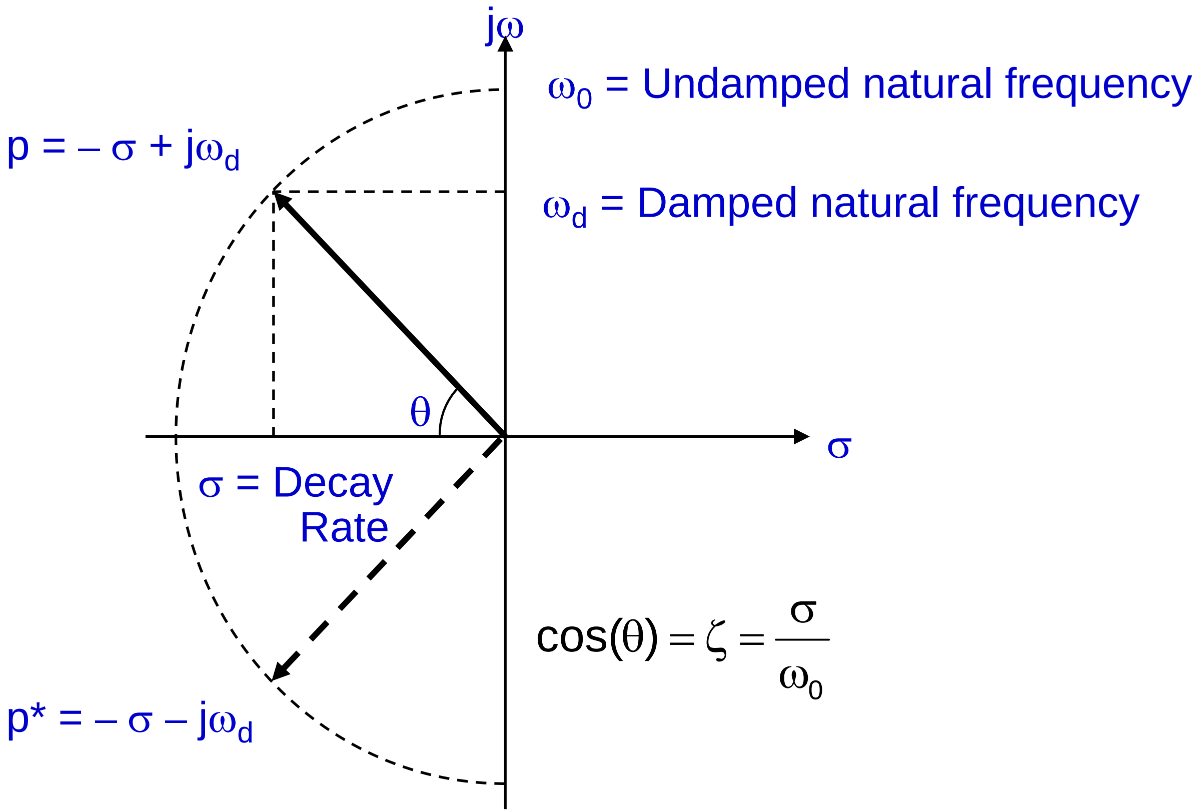
H(s) may be expanded in partial fraction form: $\quad H(s) = \dfrac{R}{s-p} + \dfrac{R^*}{s-p^*}$

where R is the residue for pole p.

$$R = H(s)\,(s-p)_{s=p} = \frac{1/m}{p-p^*} = \frac{1/m}{2j\omega_d}$$

2.2 1 DOF system

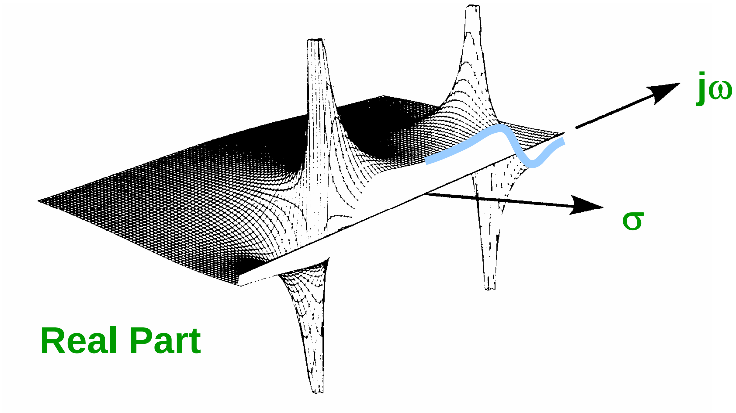
- Laplace Plane — (Pole Locations)



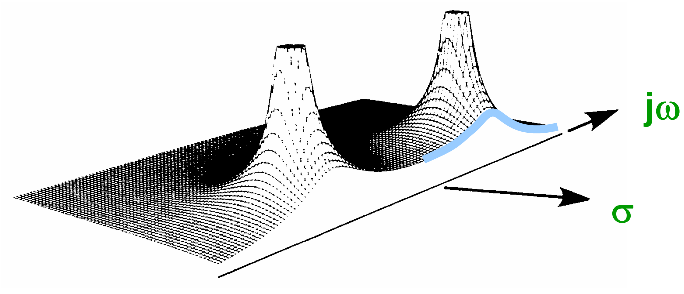
$$\cos(\theta) = \zeta = \frac{\sigma}{\omega_0}$$

2.2 1 DOF system

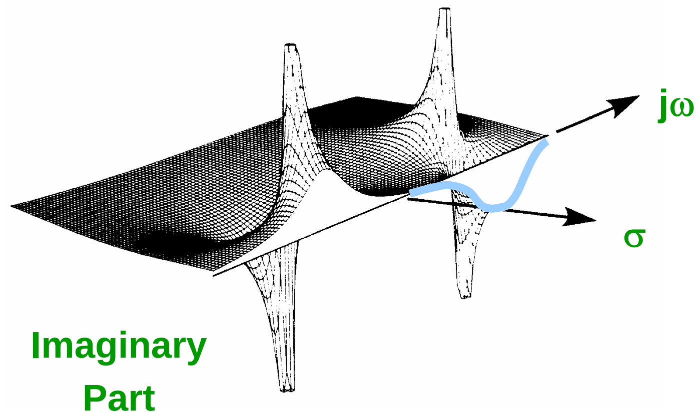
- Transfer function



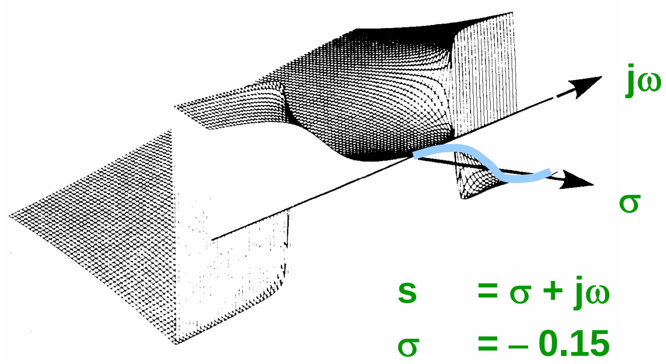
Real Part



Magnitude



Imaginary Part



Phase

$s = \sigma + j\omega$

$\sigma = -0.15$

$\omega_0 = 0.5$

Both poles are each others symmetry and thus one through pole will be taken into account, as a result of a SDOF.
The magnitude goes to infinite at the poles.

2.2 1 DOF system

- Transfer vs. Frequency Response Function

Frequency Response Function = Transfer Function along jω axis

$$H(s) = \frac{1}{(j\omega - p)(j\omega - p^*)}$$

or

$$H(s)\Big|_{s=j\omega} = \frac{R}{j\omega - (-\sigma + j\omega_d)} + \frac{R^*}{j\omega - (-\sigma - j\omega_d)}$$

2.2 1 DOF system

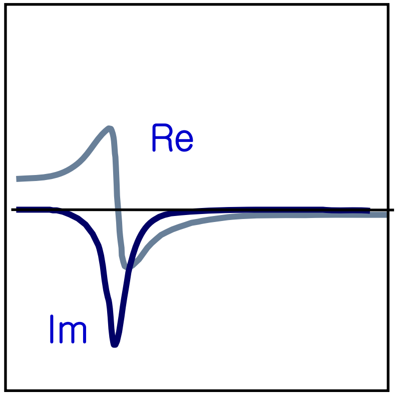
- Frequency Response Function for SDOF system

$$H(\omega) = \frac{R}{j\omega - p} + \frac{R^*}{j\omega - p^*}$$

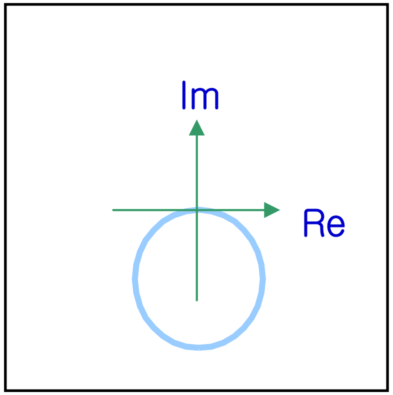
where $p = -\sigma + j\omega_d$

At resonance: $$\left|H(\omega)\right| \approx \frac{R}{\sigma}$$

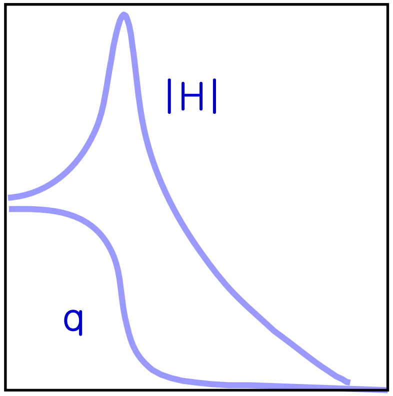
Real & Imaginary



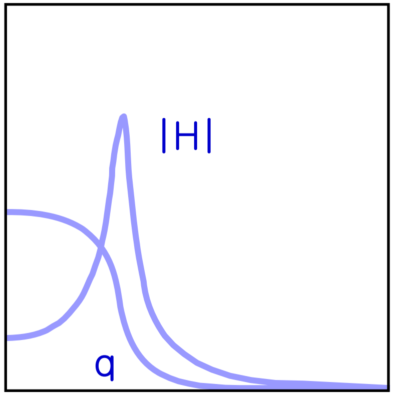
Nyquist



Log. Magnitude & Phase



Magnitude & Phase

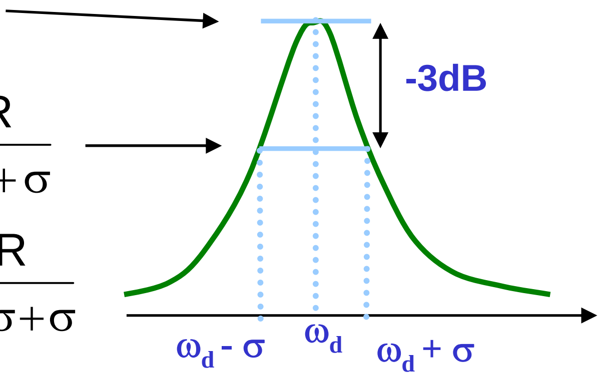


2.2 1 DOF system

- Transfer vs. Frequency Response Function

$$\boxed{H(j\omega)=\frac{R}{j\omega-(-\sigma+j\omega_d)}+\frac{R^*}{j\omega-(-\sigma+j\omega_d)}}$$

- FRF in the vicinity of ω_d: ⇨ $H\approx\dfrac{R}{j\omega-(-\sigma+j\omega_d)}$
- FRF at $\omega = \omega_d$: ⇨ $H\approx\dfrac{R}{\sigma}$
- FRF at $\omega = \omega_d + \sigma$: ⇨ $H\approx\dfrac{R}{j\sigma+\sigma}$
- FRF at $\omega = \omega_d - \sigma$: ⇨ $H\approx\dfrac{R}{-j\sigma+\sigma}$

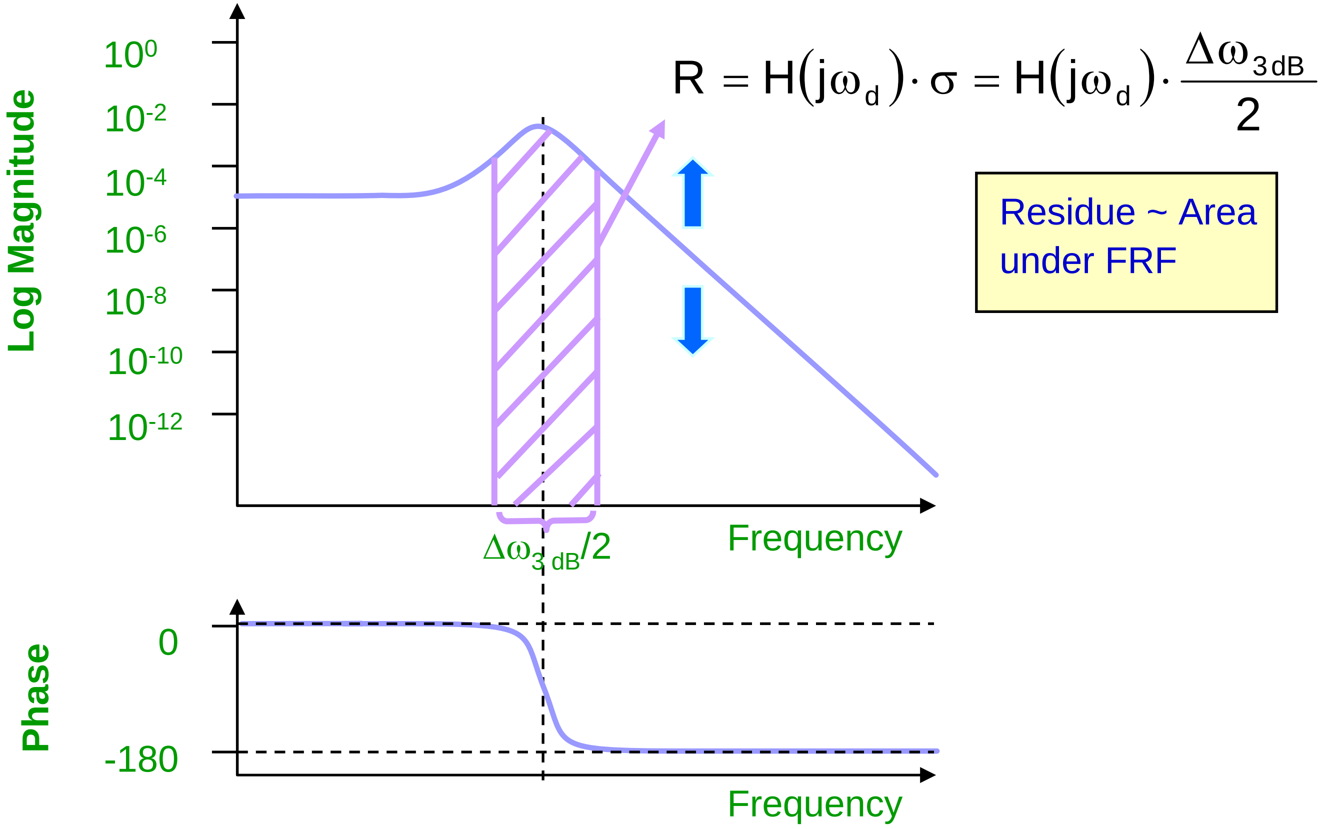


$$\left|H(j\omega_d\pm j\sigma)\right|=\frac{R}{\sqrt{2}\ \sigma}\approx\frac{H(j\omega_d)}{\sqrt{2}}$$ ⇨ **Thus $\Delta\omega_{3\ dB} = 2\sigma$**

2.2 1 DOF system

- Bandwidth of Resonance

Interpretation of Residue R = pole strength

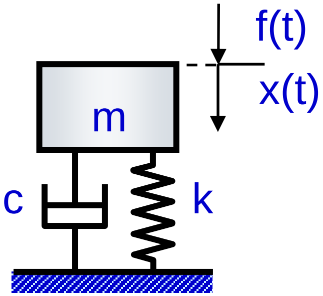


$$R = H(j\omega_d)\cdot\sigma = H(j\omega_d)\cdot\frac{\Delta\omega_{3\,dB}}{2}$$

Residue ~ Area under FRF

2.2 1 DOF system

- Impulse Response Function for SDOF Model



$$H(\omega)=\frac{X(\omega)}{F(\omega)}=\frac{1}{-\omega^2 m+j\omega c+k}=\frac{R}{j\omega-(-\sigma+j\omega_d)}+\frac{R^*}{j\omega-(-\sigma-j\omega_d)}$$

where: $R=\dfrac{1}{j2m\omega_d}$, $\omega_d=\sqrt{\dfrac{k}{m}-\dfrac{c^2}{4m^2}}$, $\sigma=\dfrac{c}{2m}$

$$\boxed{h(t)=\mathcal{F}^{-1}\{H(\omega)\}=2|R|\,e^{-\sigma t}\sin(\omega_d t)}$$

$$\frac{R}{j\omega-(-\sigma+j\omega_d)}+\frac{R^*}{j\omega-(-\sigma-j\omega_d)}$$

$$=\frac{R}{(s+\sigma)-j\omega_d}+\frac{R^*}{(s+\sigma)+j\omega_d}$$

$$=\frac{j\omega_d R-j\omega_d R^*}{(s+\sigma)^2+\omega_d^2}=\frac{2\omega_d|R|}{(s+\sigma)^2+\omega_d^2}$$

$$L[e^{-\sigma t}f(t)]=F(s+\sigma)$$

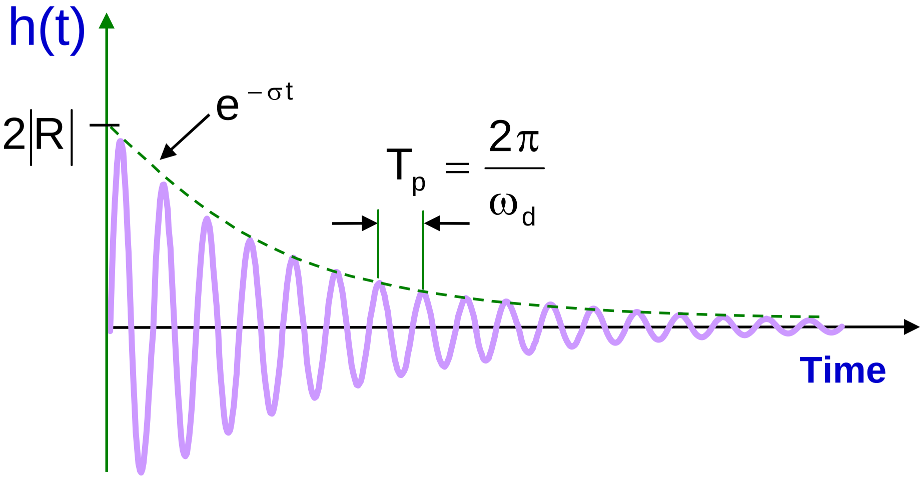
2.2 1 DOF system

- Impulse Response Function for SDOF Model

$$h(t) = \mathcal{L}^{-1}[H(s)]$$

$$h(t) = Re^{pt} + R^* e^{p^*t}$$

$$h(t) = 2|R|\, e^{-\sigma t} \cdot \sin(\omega_d \cdot t)$$



2.3 Multiple DOF system

- Impulse Response Function for SDOF Model

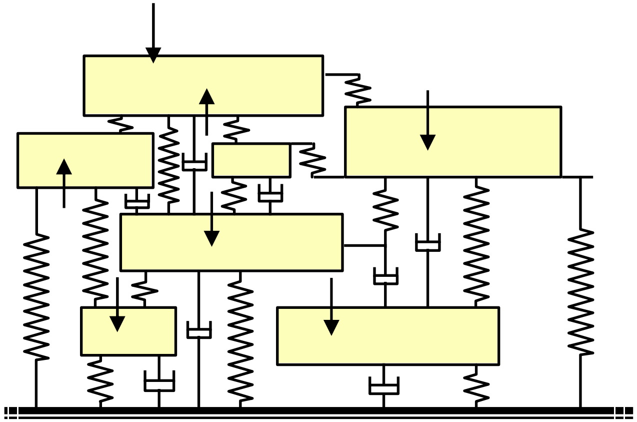
Purpose of lectures:

- To show that
  MDOF model = Σ SDOF models
- To describe the Mathematical formulation of the Modal Model

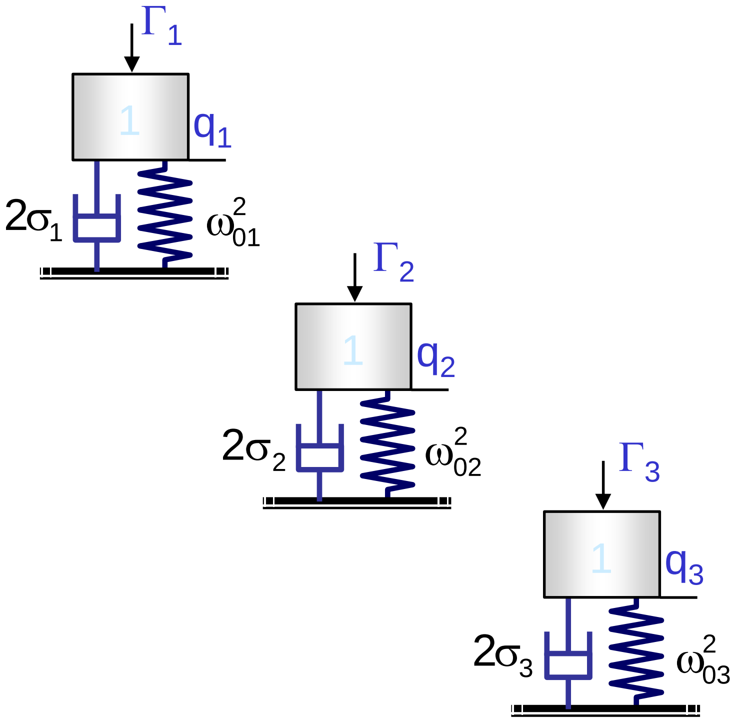
2.3 Multiple DOF system

- Multi Degree of Freedom Models (MDOF models)

Physical Coordinates = $C^{H}{}_{A}O^{S}$

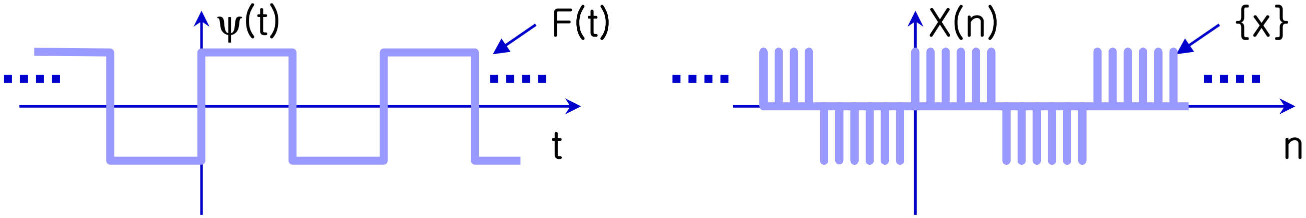


Modal Space = *Beauty*



2.3 Multiple DOF system

- Orthogonal Functions and Vectors
  - A set of functions $\{\psi_r(t)\}$ are orthogonal in $[t_1, t_2]$ if: $\int_{t_1}^{t_2} \psi_r \psi_s dt = \begin{cases} 0 & r \neq s \\ k & r = s \end{cases}$
  - Vectors are mutually orthogonal if: $\{\psi\}_x^T \{\psi\}_s = \sum_i \psi_{ir} \psi_{is} = \begin{cases} 0 & r \neq s \\ k & r = s \end{cases}$



To represent a vector {x} in the m-dimensional space spanned by the set$[\Psi]$:

$$\{x\} = c_1\{\psi\}_1 + c_2\{\psi\}_2 + \dots + c_m\{\psi\}_m \qquad \sim \qquad \{x\} = [\Psi]\{c\}$$

The constants c_r are then found:

$$\{\psi\}_r^T\{x\} = c_1\{\psi\}_r^T\{\psi\}_1 + \dots + c_r\{\psi\}_r^T\{\psi\}_r + \dots + c_m\{\psi\}_r^T\{\psi\}_m$$

$$c_r = \frac{\{\psi\}_r^T\{x\}}{\|\psi_r\|}; \quad where\ \|\psi_r\|\ is\ the\ norm\ of\ \psi_r$$

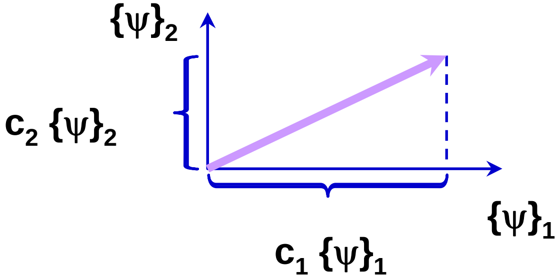
2.3 Multiple DOF system

- Transformation

$\{x\} = [\Psi]\{c\}$, is an exact presentation, as long as {x} is contained in the space spanned by the basis set [Ψ].

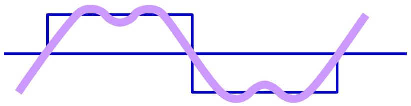
Otherwise [Ψ] {c} is an approximation!

Ex. m = 2



$\{x\} = c_1\{\psi\}_1 + c_2\{\psi\}_2$ is complete

Ex. m = 2



$\sin(\omega_0 t) + \frac{1}{3}\sin(3\omega_0 t)$

$\{F(t)\} \neq [\Psi]\{c\}$ is *not* complete

2.3 Multiple DOF system

- Eigenvalue Problem

$$[m]\{\ddot{x}\}+[k]\{x\}=\{0\}$$

Trial solution (a synchronous motion) $\{x\}=\{\phi\}\sin(\omega t+\varphi)$

$$\big[[k]-\omega^2[m]\big]\{\phi\}=\{0\}$$

Characteristic Equation:

$$\big|[k]-\omega^2[m]\big|=0 \quad \longrightarrow \quad \begin{array}{cccc} \omega_1, & \omega_2, & \omega_3, & \cdots\ \omega_n \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \begin{Bmatrix}\phi_{11}\\ \phi_{21}\\ \vdots \\ \phi_{n1}\end{Bmatrix} & \begin{Bmatrix}\phi_{12}\\ \phi_{22}\\ \vdots \\ \phi_{n2}\end{Bmatrix} & \begin{Bmatrix}\phi_{13}\\ \phi_{23}\\ \vdots \\ \phi_{n3}\end{Bmatrix} & \begin{Bmatrix}\phi_{1n}\\ \phi_{2n}\\ \vdots \\ \phi_{nn}\end{Bmatrix} \end{array}$$

2.3 Multiple DOF system

- Free vibration response

$$\left[m\right]\left\{\ddot{x}\right\}+\left[k\right]\left\{x\right\}=\left\{0\right\}$$

$$\left\{x(t)\right\}=c_1\left\{\phi\right\}_1\sin(\omega_1 t+\varphi_1)+c_2\left\{\phi\right\}_2\sin(\omega_2 t+\varphi_2)+\cdots+c_3\left\{\phi\right\}_n\sin(\omega_n t+\varphi_n)$$

[I.C.] ⇒ $c_1\,,\ \varphi_1\,,\ c_2\,,\ \varphi_2\,,\ \cdots\,,\ c_n\,,\ \varphi_n$

- Forced response
  - Direct Inverse Method
  - Modal Decoupling Method

$$\left[m\right]\left\{\ddot{x}\right\}+\left[k\right]\left\{x\right\}=\left\{F\right\}\sin\omega t$$

Trial solution $\left\{x\right\}=\left\{X\right\}\sin\omega t$

$$\left[\left[k\right]-\omega^2[m]\right]\left\{X\right\}=\{F\}$$

$$\left\{X\right\}=\left[\left[k\right]-\omega^2[m]\right]^{-1}\{F\}=[H]\{F\}$$

2.3 Multiple DOF system

- Mode Shapes and Orthogonality

The most important property of Mode Shapes is the orthogonality!

Proof: $$[m]\{\ddot{x}\}+[k]\{x\}=\{0\}$$

Fourier Transform, evaluated for $\omega = \omega_{or}$:

$$-\omega_{or}^2[m]\{\phi\}_r+[k]\{\phi\}_r=\{0\}$$

Pre-multiply by $\{\phi\}_s^T$

$$-\omega_{or}^2\{\phi\}_s^T[m]\{\phi\}_r+\{\phi\}_s^T[k]\{\phi\}_r=0 \qquad [1]$$

or $$-\omega_{os}^2\{\phi\}_r^T[m]\{\phi\}_s+\{\phi\}_r^T[k]\{\phi\}_s=0$$

Transposed:

$$-\omega_{os}^2\{\phi\}_s^T[m]^T\{\phi\}_r+\{\phi\}_s^T[k]^T\{\phi\}_r=0 \qquad [2]$$

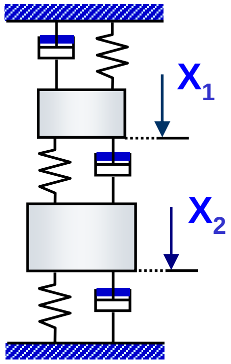
Subtract [1]- [2]:

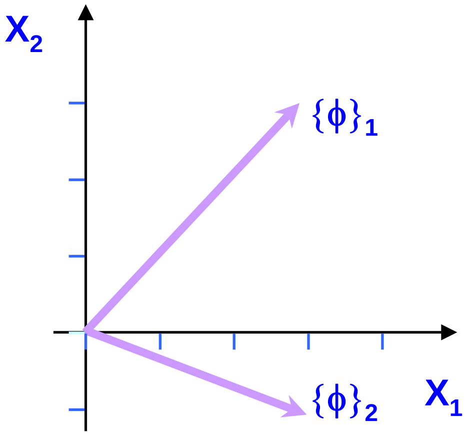
$$(\omega_{os}^2-\omega_{or}^2)\{\phi\}_s^T[m]\{\phi\}_r=0$$

For modes with $\omega_{os}\neq\omega_{or}$

$$\{\phi\}_s^T[m]\{\phi\}_r=\begin{cases}\mathbf{0\ for\ } s\neq r\\ \mathbf{M_r\ for\ } s=r\end{cases}$$

Also for [k], and [c] = α [m] + β [k]





2.3 Multiple DOF system

- Utilizing the Mode Shape Orthogonality

$$[\Phi]=\left[\{\phi\}^1\{\phi\}^2\cdots\{\phi\}^m\right]$$

MODAL MATRIX

$$[\Phi]^T[m][\Phi]=\left\lceil M_r \right\rfloor$$

$$[\Phi]^T[k][\Phi]=\left\lceil K_r \right\rfloor$$

$$\left([\Phi]^T[c][\Phi]=\left\lceil C_r \right\rfloor\right)$$

Uncoupling of equations by the Modal Transformation

$$\{x\}=[\Phi]\{q\}$$

physical coordinates → **modal coordinates**

$$[m]\{\ddot{x}\}+[c]\{\dot{x}\}+[k]\{x\}=\{f\}$$

Substitute $\{x\}$ and pre-multiply by $[\Phi]^T$

$$\left\lceil M_r \right\rfloor\{\ddot{q}\}+\left\lceil C_r \right\rfloor\{\dot{q}\}+\left\lceil K_r \right\rfloor\{q\}=[\Phi]^T\{f\}$$

- Unit Modal Mass Scaling

$$[\Phi]^T[m][\Phi]=\left\lceil I \right\rfloor=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$\Rightarrow\left\lceil I \right\rfloor\{\ddot{q}\}+\left\lceil 2\sigma_r \right\rfloor\{\dot{q}\}+\left\lceil \omega_0^2 \right\rfloor\{q\}=\{\Gamma(t)\}$$

2.3 Multiple DOF system

- FRF of MDOF System

$$\left[{}^{\diagdown} \mathrm{M}_r {}_{\diagdown}\right]\{\ddot{\mathrm{q}}\}+\left[{}^{\diagdown} \mathrm{C}_r {}_{\diagdown}\right]\{\dot{\mathrm{q}}\}+\left[{}^{\diagdown} \mathrm{K}_r {}_{\diagdown}\right]\{\mathrm{q}\}=[\Phi]^{\mathrm{T}}\{\mathrm{f}\}$$

$$\{f\}=\{F\}\sin\omega t \qquad \{q\}=\{Q\}\sin\omega t$$

$$\{Q\}=\left[{}^{\diagdown} \frac{1}{K_r-\omega^2 M_r+j\omega C_r} {}_{\diagdown}\right][\Phi]^T\{F\}$$

$$\{\mathrm{x}\}=[\Phi]\{\mathrm{q}\}$$

$$\{X\}=[\Phi]\{Q\}=[\Phi]\left[{}^{\diagdown} \frac{1}{K_r-\omega^2 M_r+j\omega C_r} {}_{\diagdown}\right][\Phi]^T\{F\}$$

2.3 Multiple DOF system

$$[H] = [\Phi]\begin{bmatrix} \ddots & & \\ & \dfrac{1}{K_r - \omega^2 M_r + j\omega C_r} & \\ & & \ddots \end{bmatrix}[\Phi]^T$$

$$H_{ij}(\omega) = \sum_{r=1}^{n} \frac{\phi_{ir}\phi_{jr}}{K_r - \omega^2 M_r + j\omega C}$$

$$\begin{Bmatrix} X_1(\omega) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ X_n(\omega) \end{Bmatrix} = \begin{bmatrix} H_{11}(\omega) & H_{12}(\omega) & \cdot & \cdot & H_{1n}(\omega) \\ H_{21}(\omega) & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ H_{n1}(\omega) & \cdot & \cdot & \cdot & H_{nn}(\omega) \end{bmatrix} \begin{Bmatrix} F_1(\omega) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ F_n(\omega) \end{Bmatrix}$$

2.3 Multiple DOF system

$$H_{ij}(\omega)=\sum_{r=1}^{m}\frac{\phi_{ir}\phi_{jr}}{K_r-\omega^2 M_r+j\omega C_r}=\sum_{r=1}^{m}\frac{R_{ijr}}{j\omega-\left(j\omega_{dr}-\sigma_r\right)}+\frac{R_{ijr}^{*}}{j\omega-\left(-j\omega_{dr}-\sigma_r\right)}$$

R_{ijr} = a_r ϕ_{ir} ϕ_{ir} residue for mode # r

ω_{dr} = Damped natural frequency for mode # r

σ_r = Decay rate for mode # r

m = Number of modes in model

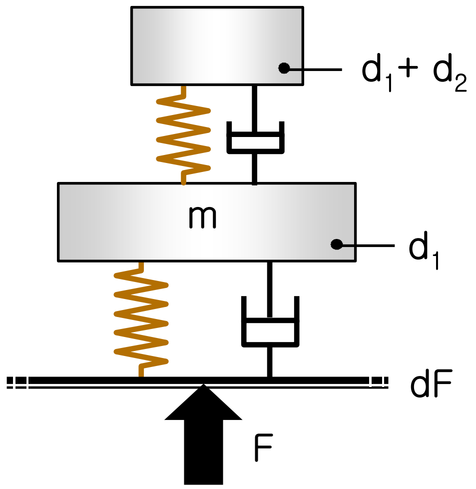
n = Number of DOF’s (x, y, z, θ_x, θ_y, θ_z)

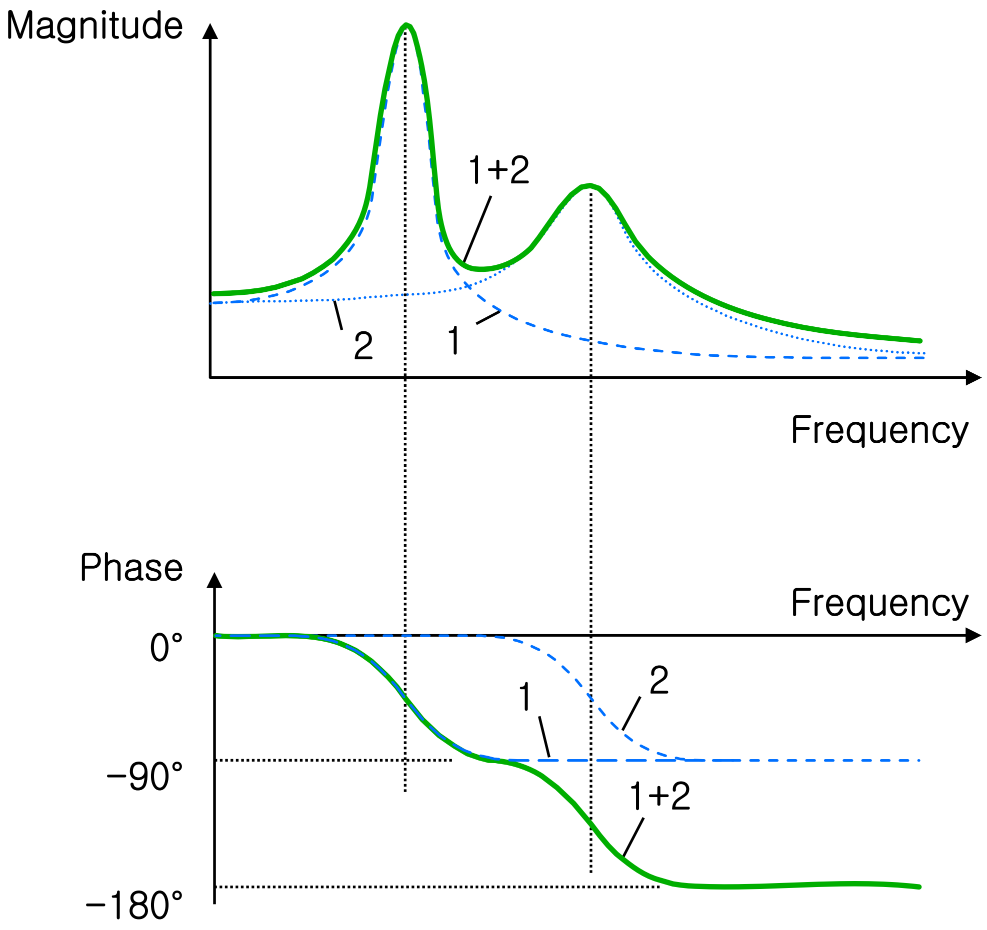
$$H(\omega)=\frac{X(\omega)}{F(\omega)}=\frac{1}{-\omega^2 m+j\omega c+k}=\frac{R}{j\omega-(-\sigma+j\omega_d)}+\frac{R^*}{j\omega-(-\sigma-j\omega_d)}$$

where: $R=\dfrac{1}{j2m\omega_d}$, $\omega_d=\sqrt{\dfrac{k}{m}-\dfrac{c^2}{4m^2}}$, $\sigma=\dfrac{c}{2m}$

2.3 Multiple DOF system

- Responses Combine



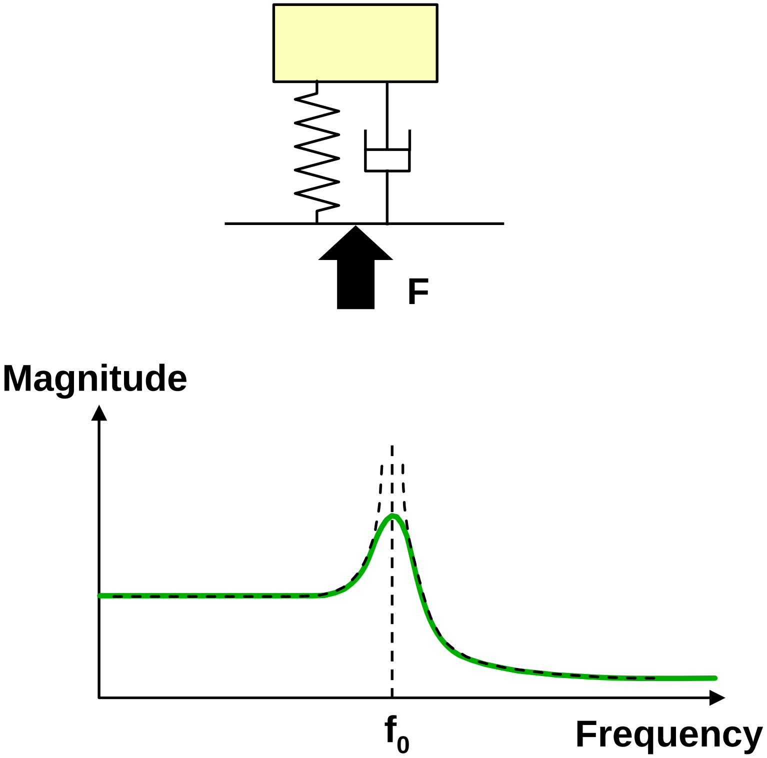


2.3 Multiple DOF system

- Response Models

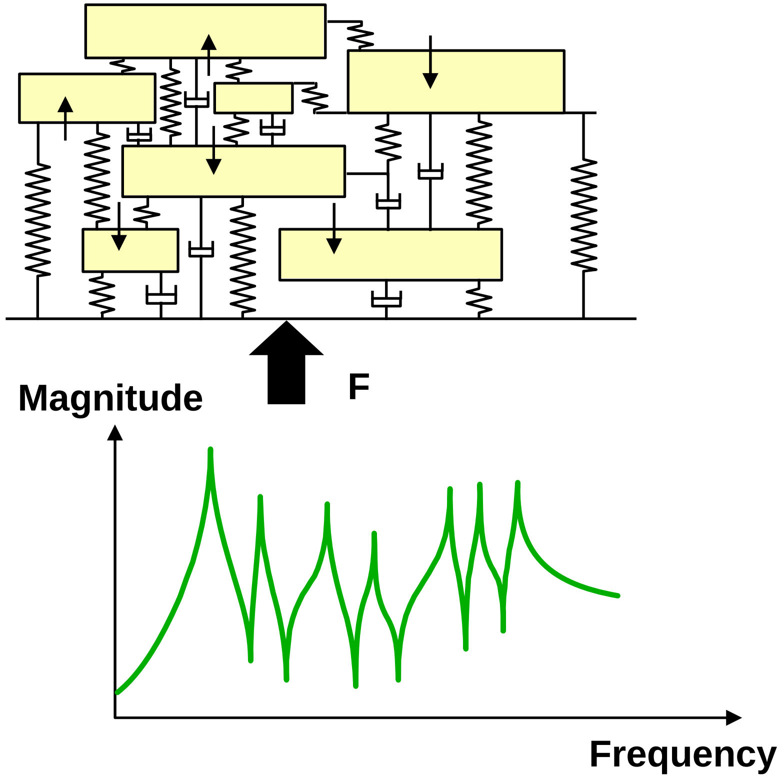
Single Degree of Freedom

SDOF



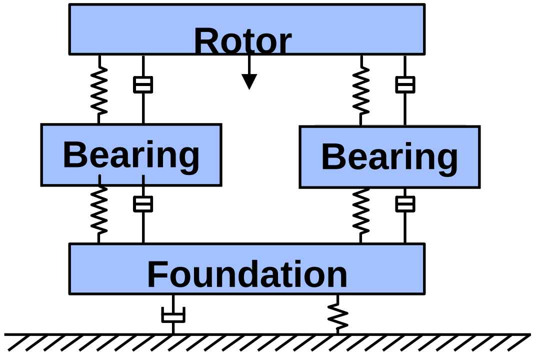
Multi Degree of Freedom

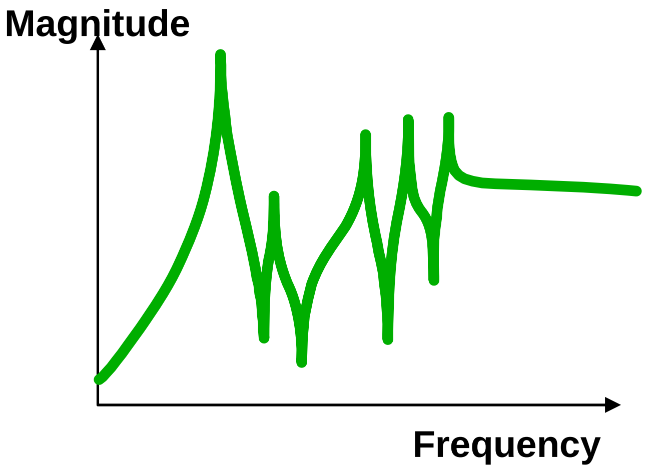
MDOF

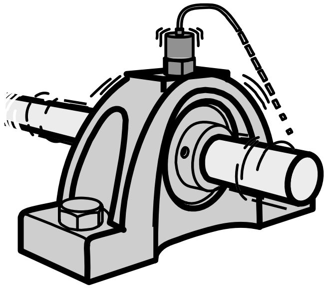


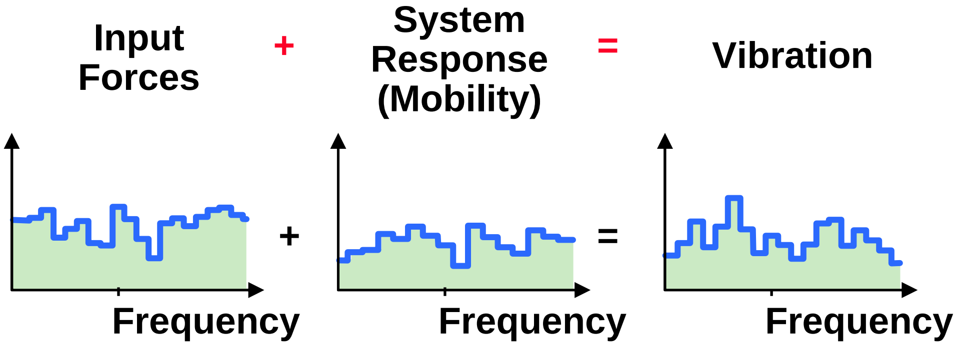
2.3 Multiple DOF system

- “Real–world” Response









Forces caused by

- Imbalance
- Shock
- Friction
- Acoustic

Structural Parameters:

- Mass
- Stiffness
- Damping

Vibration Parameters:

- Acceleration
- Velocity
- Displacement

2.3 Multiple DOF system

- The Modal Model Space

- $\{x\} = [\Phi]\,\{q\}$ is an exact presentation as long as $\{x\}$ is contained in the Modal Space by $[\Phi]$

Possible Problems

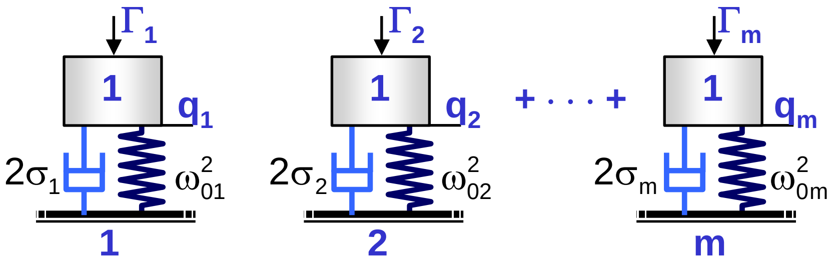
- Too few modes in $[\Phi]$ (Truncation)
- Lack of orthogonality (Undersampling)
- Non-proportional damping
- Non-linearities

- Interpretation :

$$\{\ddot{q}\}+[2\sigma_r]\{\dot{q}\}+[\omega_{0r}^2]\{q\}=\{\Gamma(t)\}$$

$$\ddot{q}_r + 2\sigma_r\,\dot{q}_r + \omega_{0r}^2\,q_r = \Gamma(t)$$

or $$\left(-\omega^2 + 2j\sigma_r\omega + \omega_{0r}^2\right)Q_r = \Gamma(\omega)$$



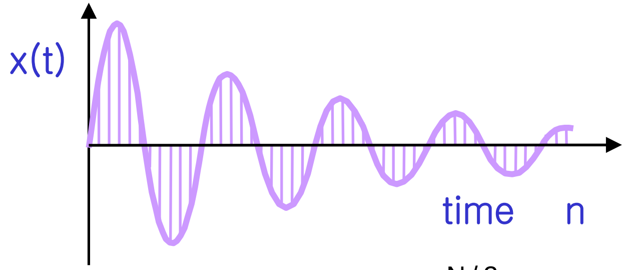
- Generalized Force

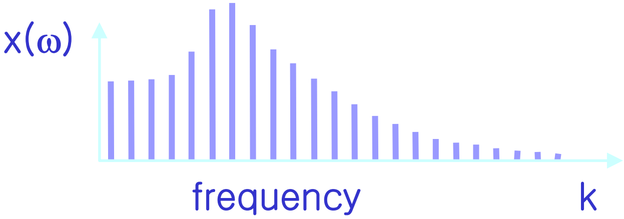
$$\{\Gamma(t)\}=[\Phi]^T\{f\}\;;\quad \Gamma(t)=\sum_{k=1}^{n}\phi_{kr}\,f_k$$

2.3 Multiple DOF system

- Fourier – vs. Modal Transformation

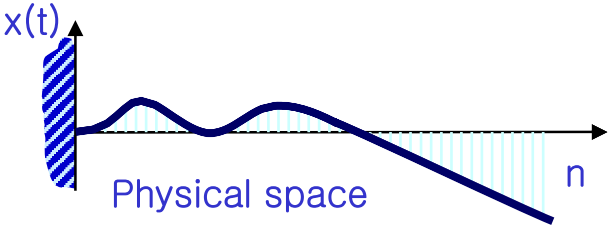
Fourier

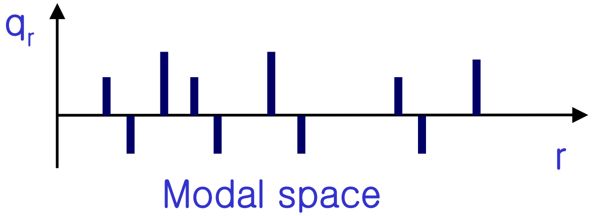




$x(n) = \sum_{k}^{N/2} X(k)\, e^{j\frac{nk}{N}}$ or $x(t) = \left[e^{j\omega t}\right]\left\{X(\omega)\right\} = \sum_{n=1}^{N/2} X_n \sin(\omega_n t)$

Modal





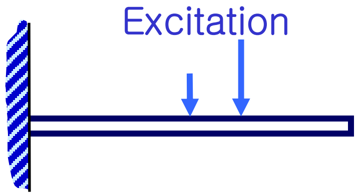
$x(n) = \sum_{r=1}^{m} \phi_{nr} \cdot q_r$ or $\{x\} = [\Phi]\{q\} = \sum_{r=1}^{m} \{\phi\}_r \cdot q_r$

Imagine:

2.3 Multiple DOF system

- Example of Solution in Modal Coordinates

$$\text{Excitation } \{f\} = \begin{Bmatrix} 0 \\ \cdot \\ \cdot \\ 25 \\ 75 \\ \cdot \\ 0 \end{Bmatrix} \cdot g(t), \qquad g(t) = \sin(30 \cdot t)$$



Structure: 300 DOF, 5 modes

$$[\Phi]_{300,5} \quad \left[{}^{\diagdown}p_{\diagdown}\right]_5$$

Solve in Modal Space: 5 uncoupled equations

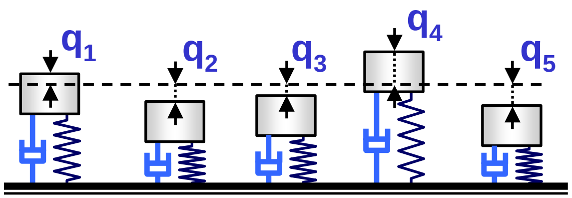
$$\ddot{q}_r(t) + 2\sigma_r\,\dot{q}(t) + \omega_{0r}^2\,q(t) = \{\Phi\}_r^T\,\{f\} = \Gamma(t)$$

or in frequency domain =

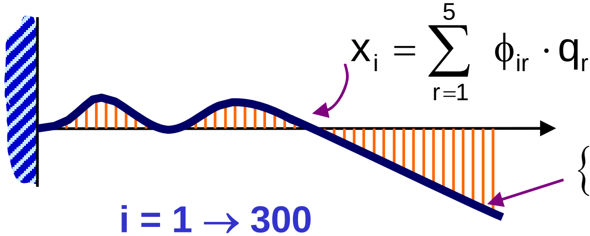
$$Q_r(\omega) = \frac{\Gamma(\omega)}{(\omega)_{0r}^2 - \omega^2 + 2j\sigma_r\omega} \Rightarrow \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{Bmatrix}$$

Transform back to Physical Coordinates:

$$\{x\} = [\Phi]\{q\} = \{\phi\}_1\,q_1 + \{\phi\}_2\,q_2 + \cdots + \{\phi\}_5\,q_5$$



$r = 1 \rightarrow 5$



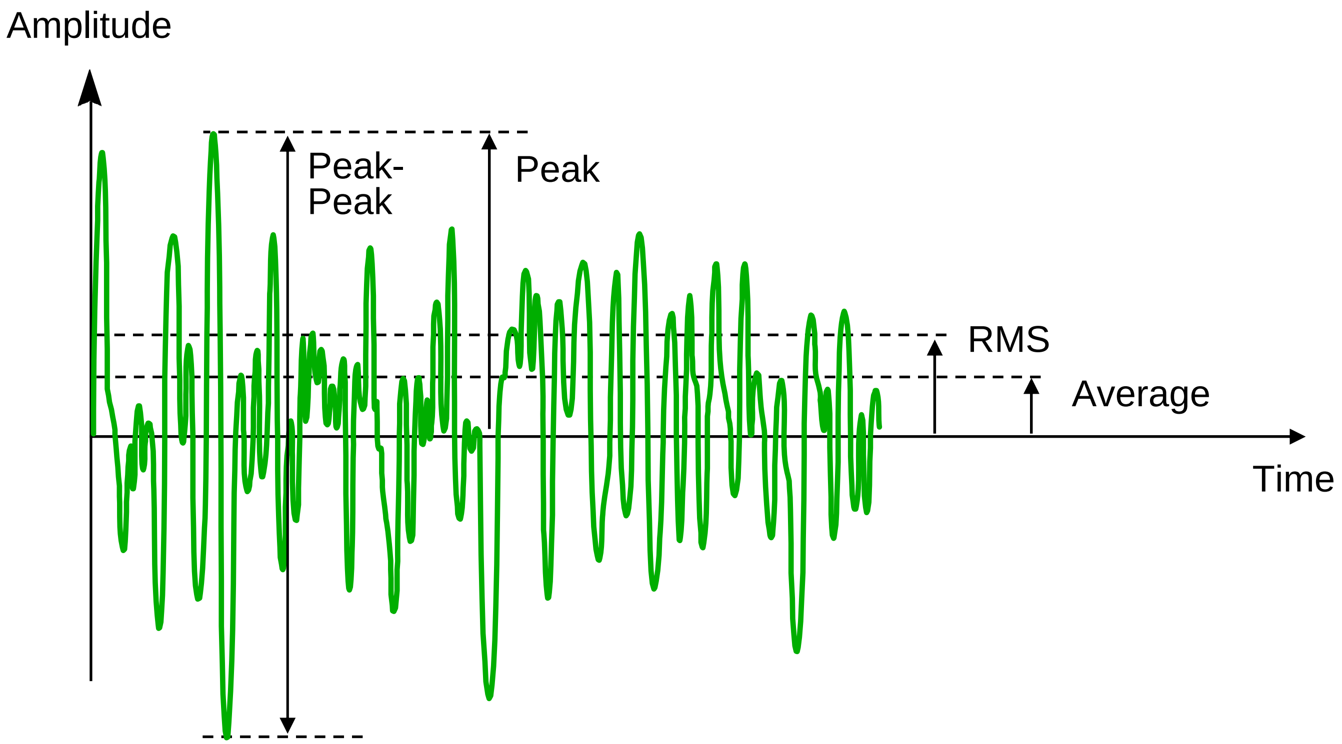
$$x_i = \sum_{r=1}^{5} \phi_{ir} \cdot q_r$$

$$\{x\} = \sum_{r=1}^{5} \{\phi\}_r\,q_r$$

$i = 1 \rightarrow 300$

2.4 Vibration Measurement

- Time Signal Descriptors



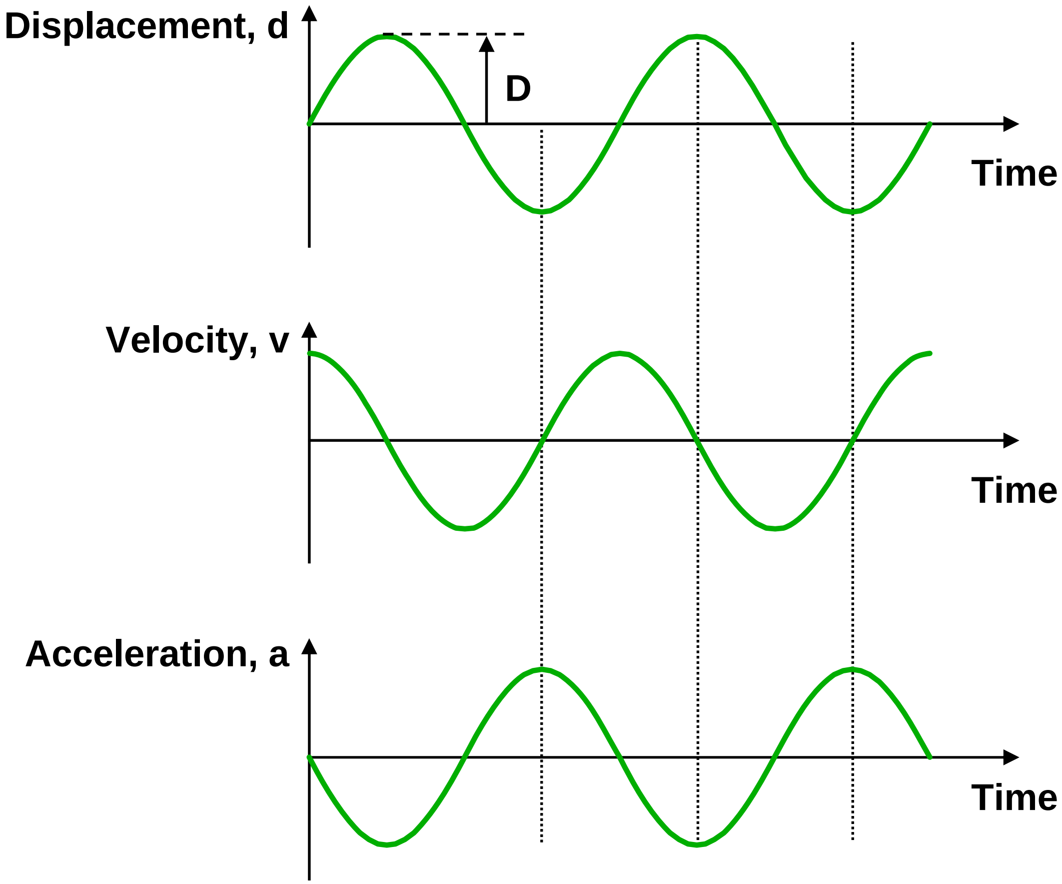
$$\mathrm{RMS} = \sqrt{\frac{1}{T}\int_0^T x^2(t)\,dt}$$

$$\mathrm{Average} = \frac{1}{T}\int_0^T |x(t)|\,dt$$

Crest Factor : $\dfrac{\textbf{Peak}}{\textbf{RMS}}$

2.4 Vibration Measurement

- Conversion from Displacement to Acceleration



$d = D \sin \omega t$

$\boxed{d = D}$

$v = \frac{dd}{dt} = D\omega \cos \omega t$

$\boxed{v = D\omega = D2\pi f}$

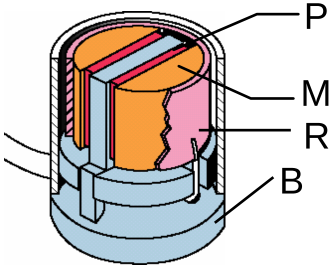
$v = \frac{d^2d}{dt^2} = D\omega^2 \sin \omega t$

$\boxed{a = D\omega^2 = D4\pi^2 f^{\,2}}$

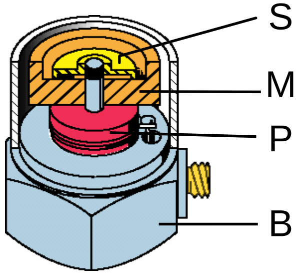
2.4 Vibration Measurement

- Types of Accelerometers

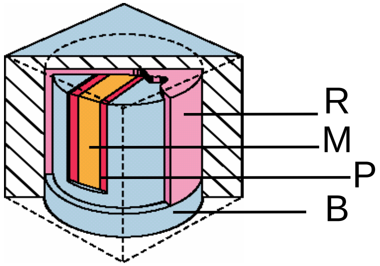
Planar Shear



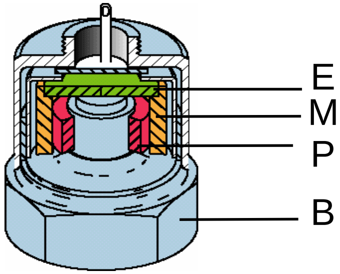
Centre-mounted Compression



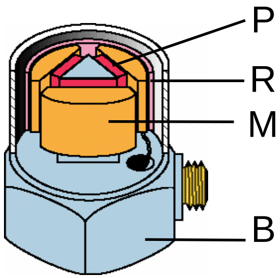
ThetaShear ®



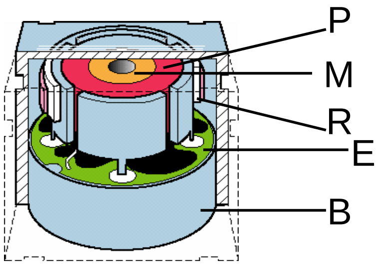
Annular Shear



Delta Shear ®



OrthoShear ®



P: Piezoelectric Elements

E: Built-in Electronics

S: Spring

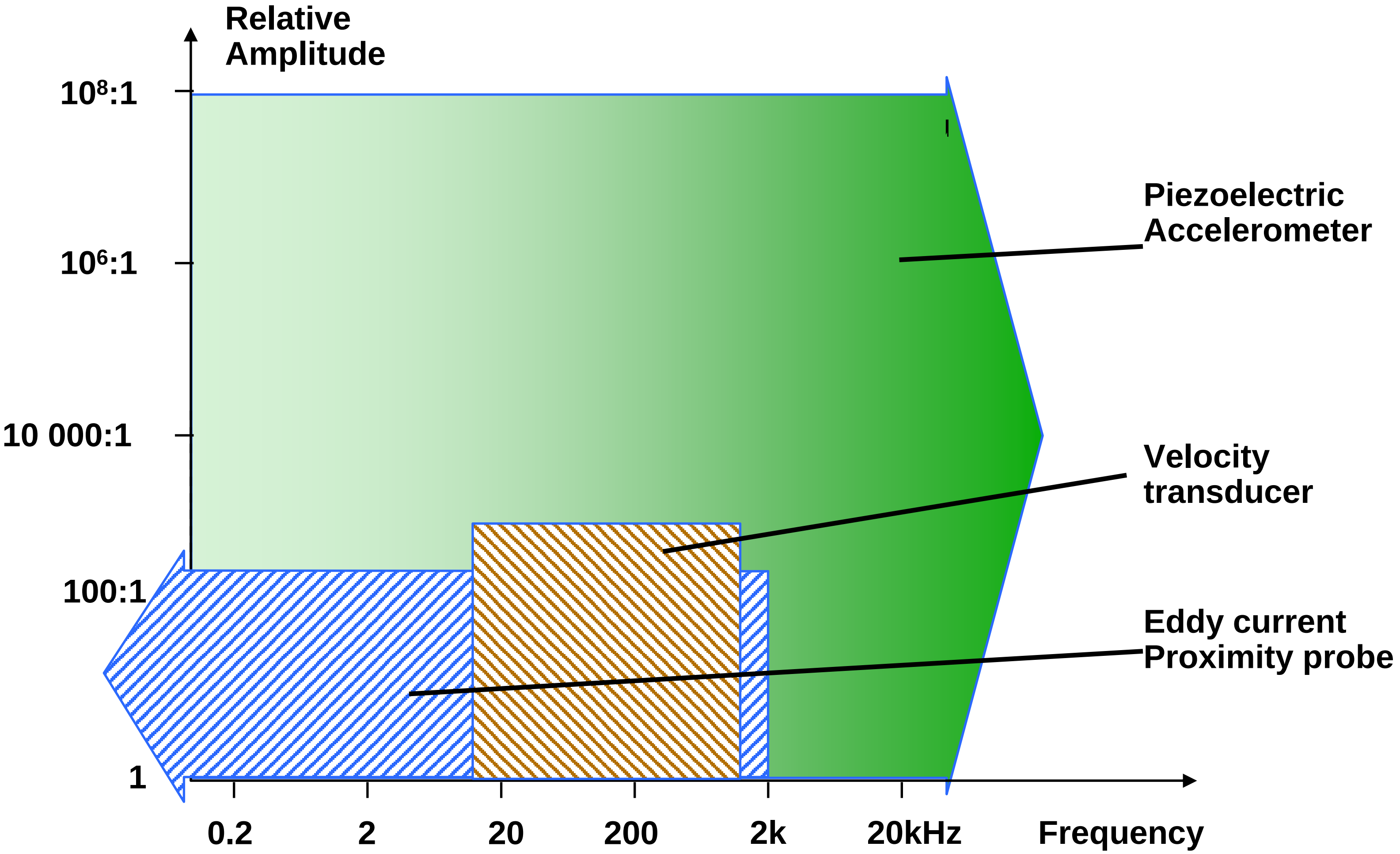
R: Clamping Ring

B: Base

M: Seismic Mass

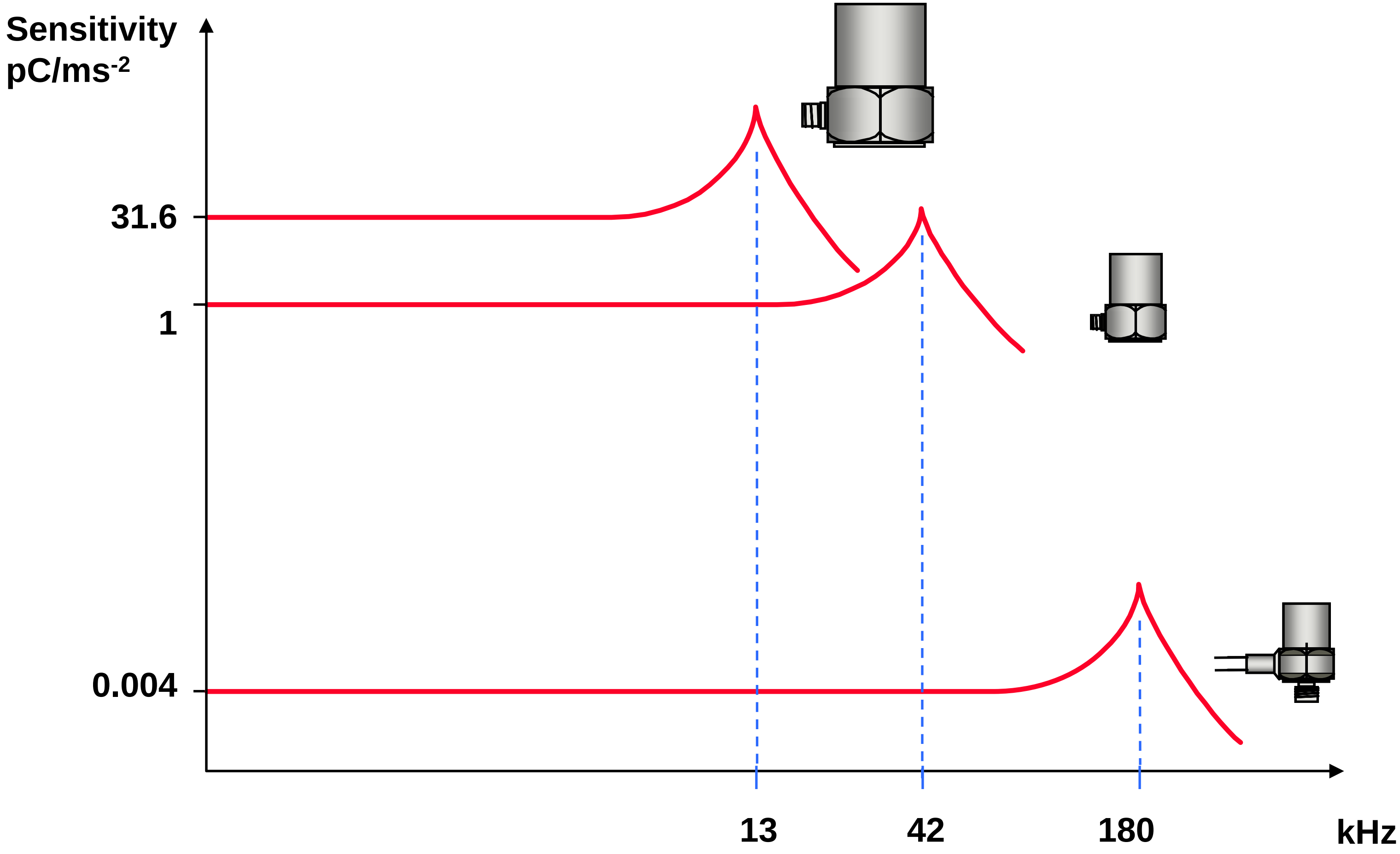
2.4 Vibration Measurement

- Operational Range of Vibration Transducers



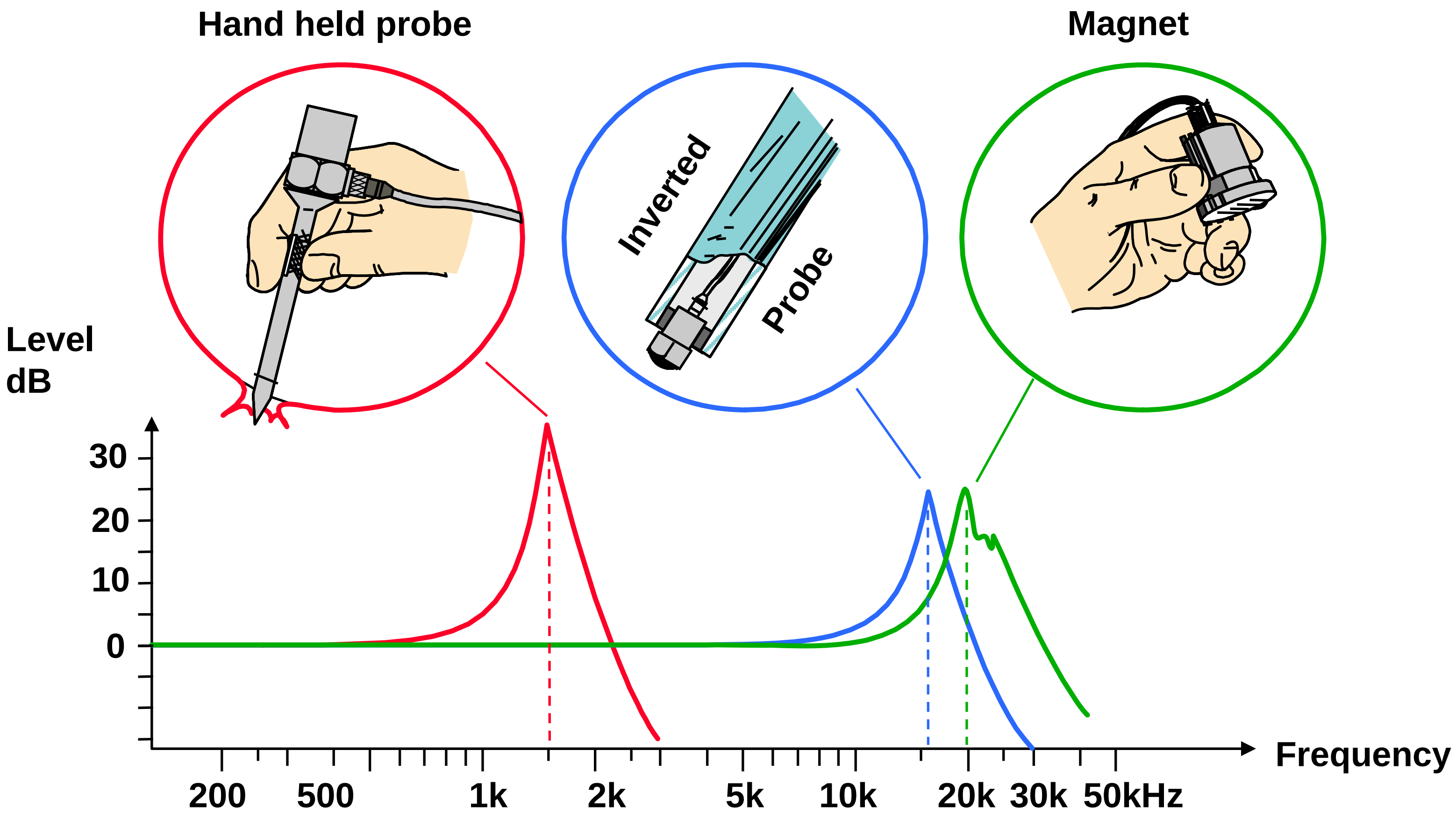
2.4 Vibration Measurement

- Sensitivity and Frequency Range



2.4 Vibration Measurement

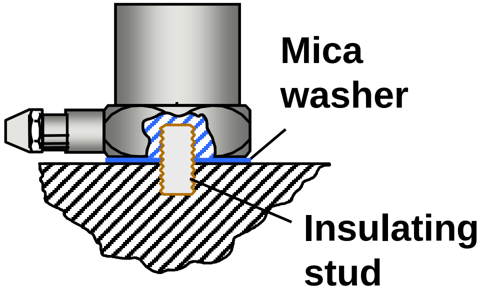
- Accelerometer Mounting — Handheld

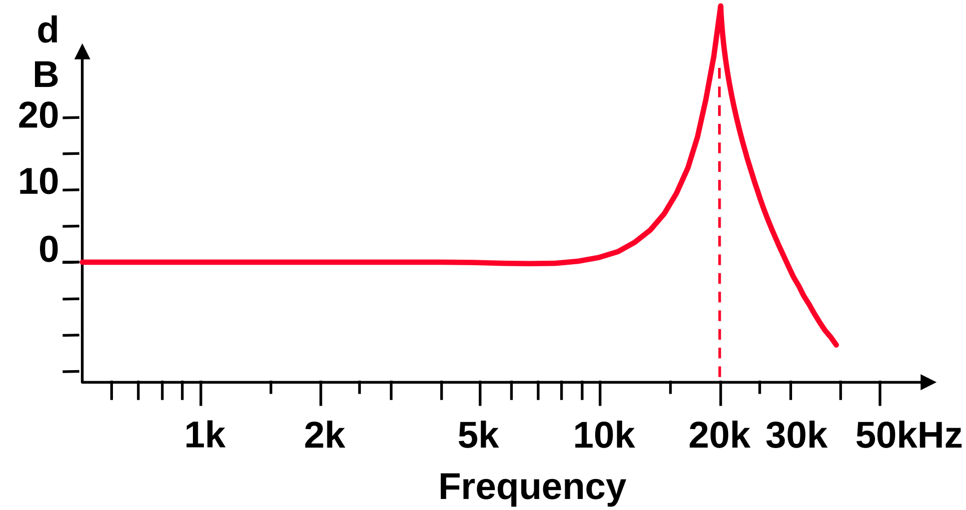


2.4 Vibration Measurement

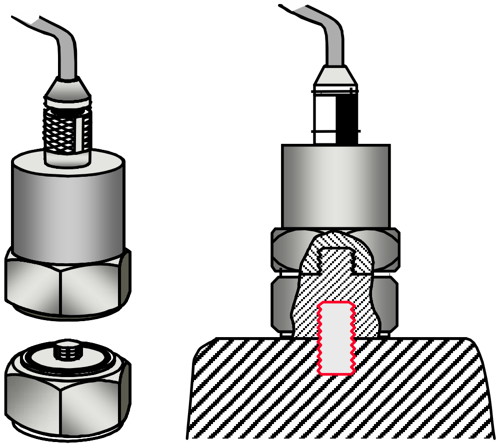
- Isolating the Accelerometer

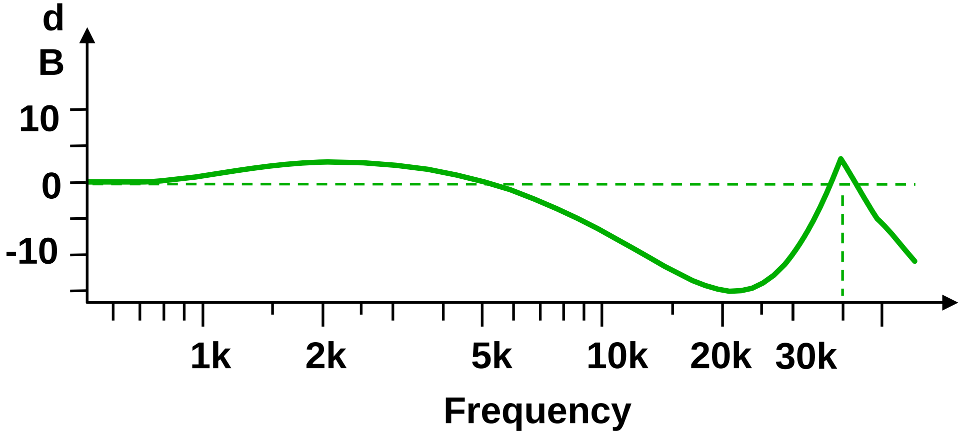
Electrical
(Prevention of ground loops)





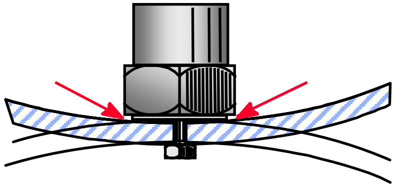
Mechanical Filter
(Protection against high shocks)



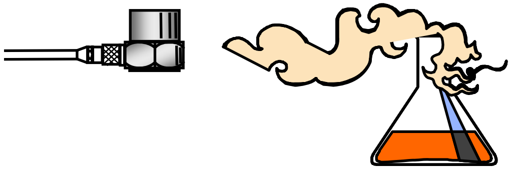


- Environmental Effects

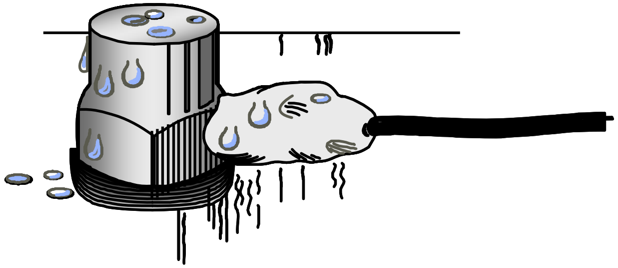
- **Base Strain**



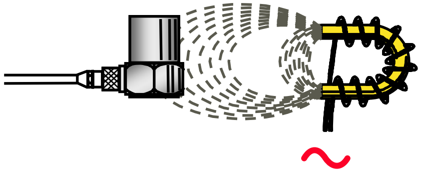
- **Corrosive substances**



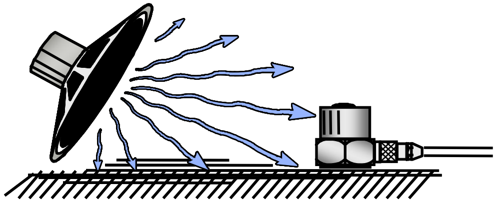
- **Humidity**



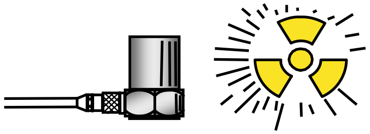
- **Magnetic fields**



- **Acoustic noise**

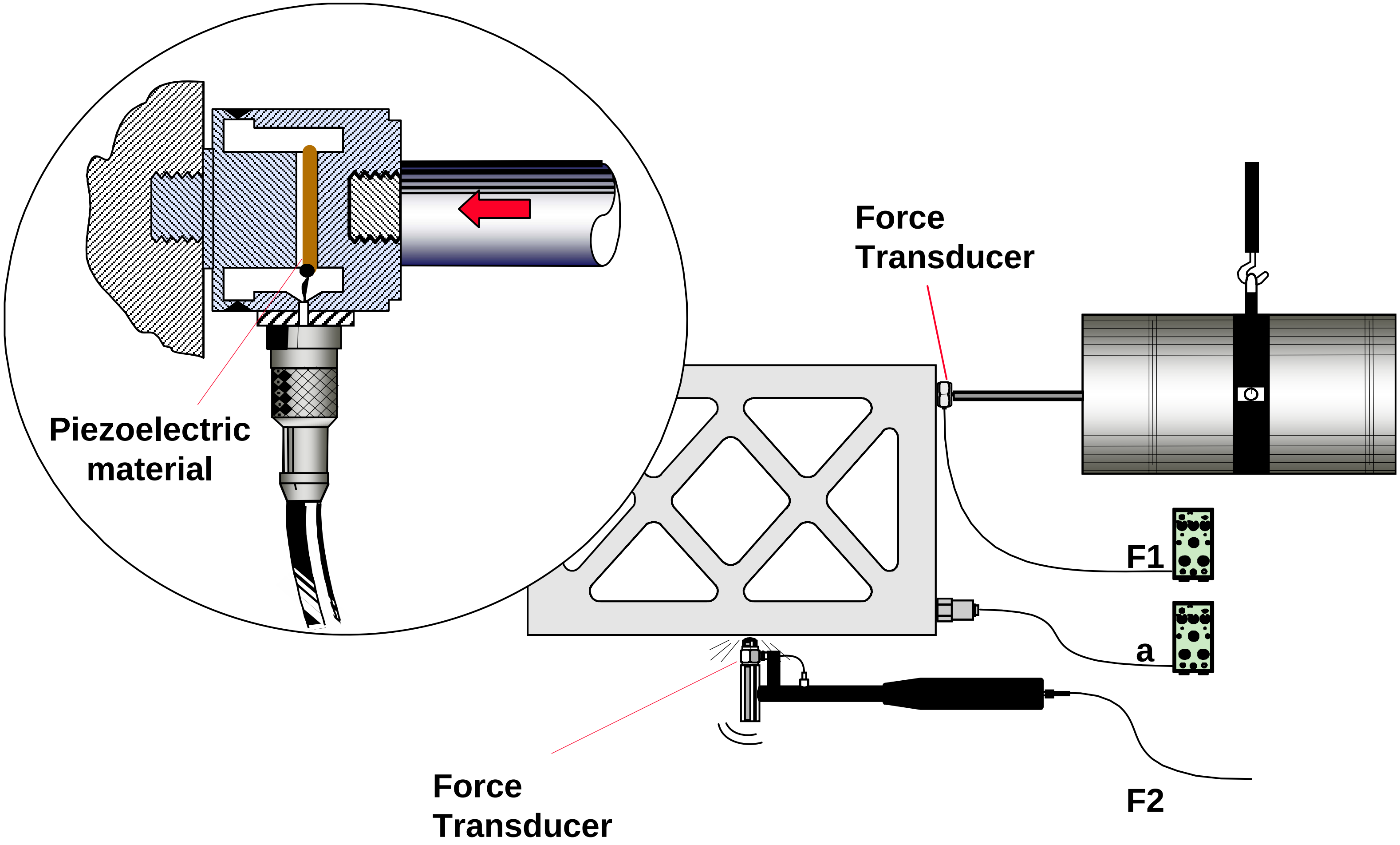


- **Nuclear radiation**



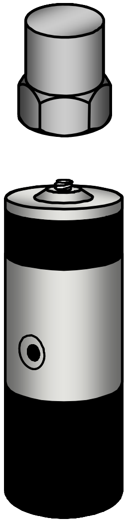
2.4 Vibration Measurement

- Force Transducer



2.4 Vibration Measurement

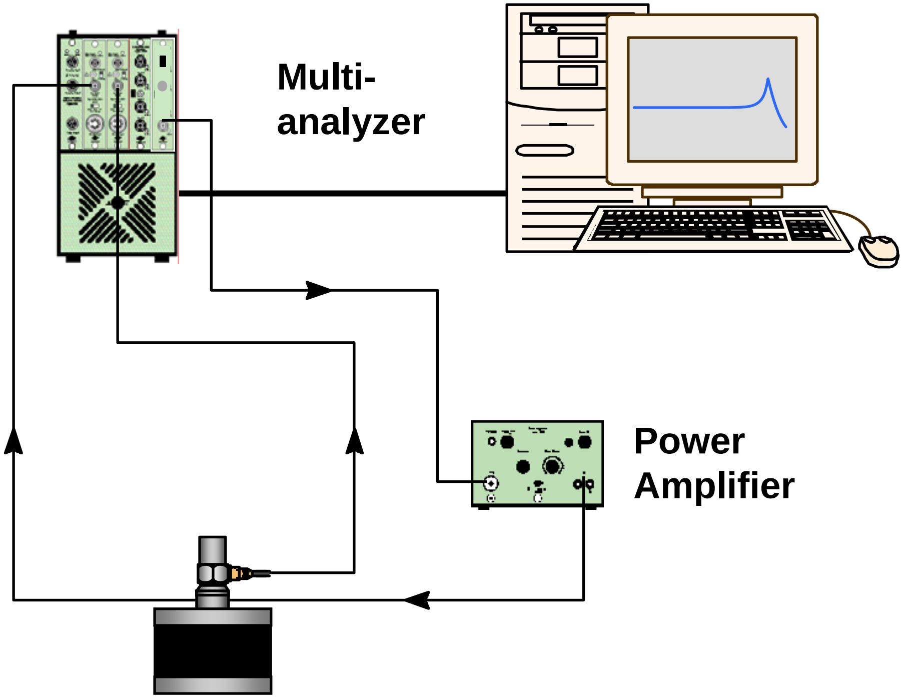
- Accelerometer Check



Frequency = 159.2 Hz

ω = 1000 rad/sec

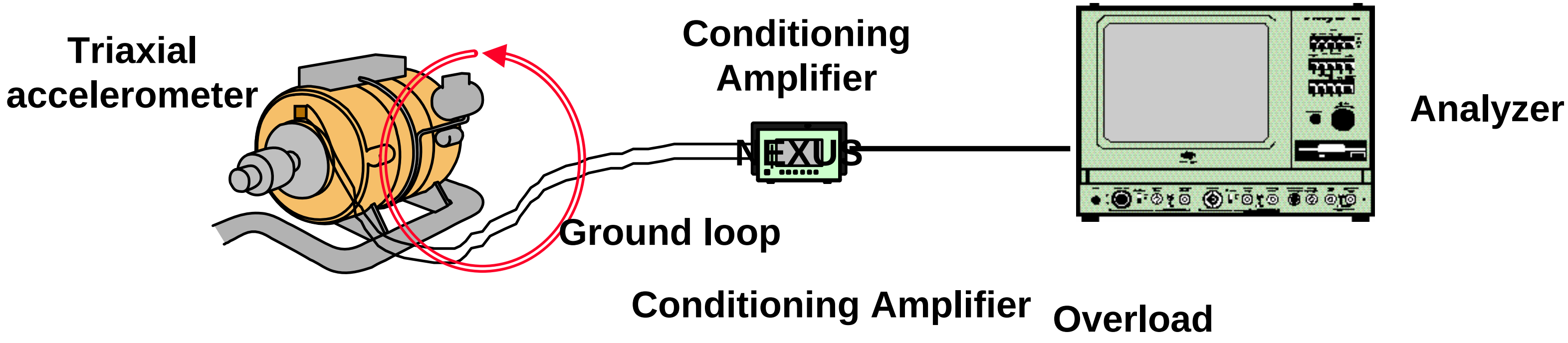
Acceleration = 10 ms^{-2}



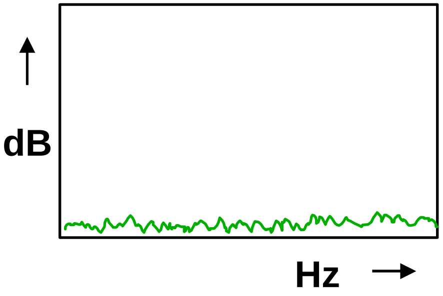
**Calibration Exciter with built-in
or external reference accelerometer**

2.4 Vibration Measurement

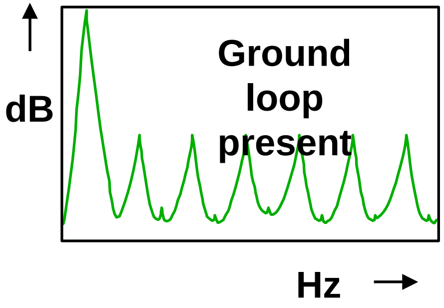
- Ground Loop Problem and Solution



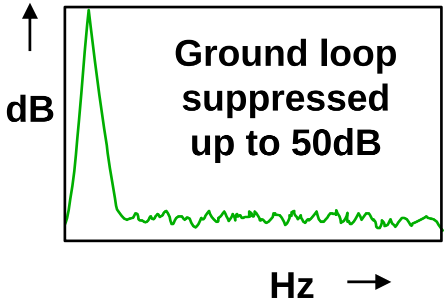
- **Inherent noise in system**



- **Signal present**
- **Machine operating**

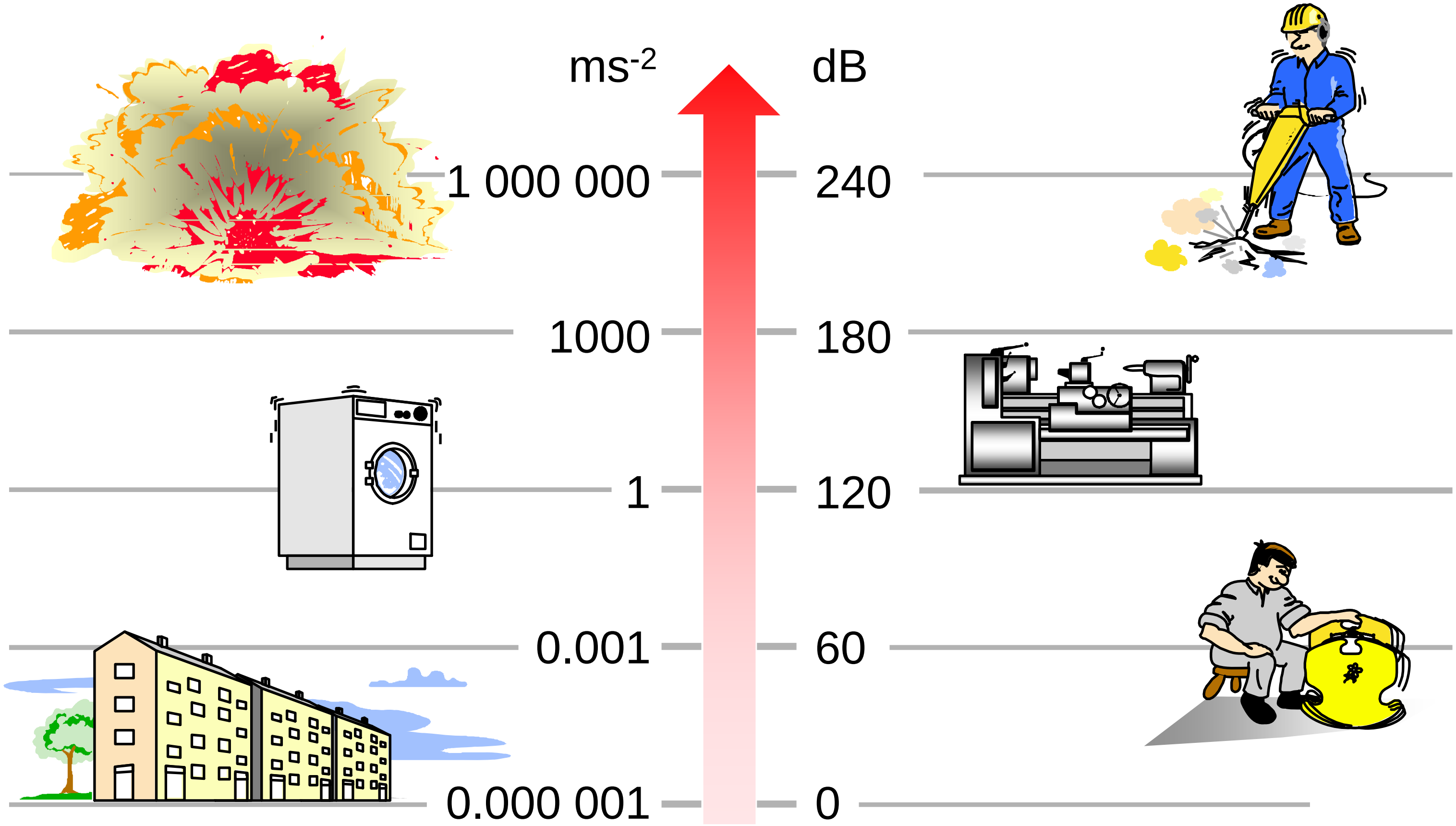


- **Signal present.**
- **Machine operating**
- **Input set to "FLOATING"**



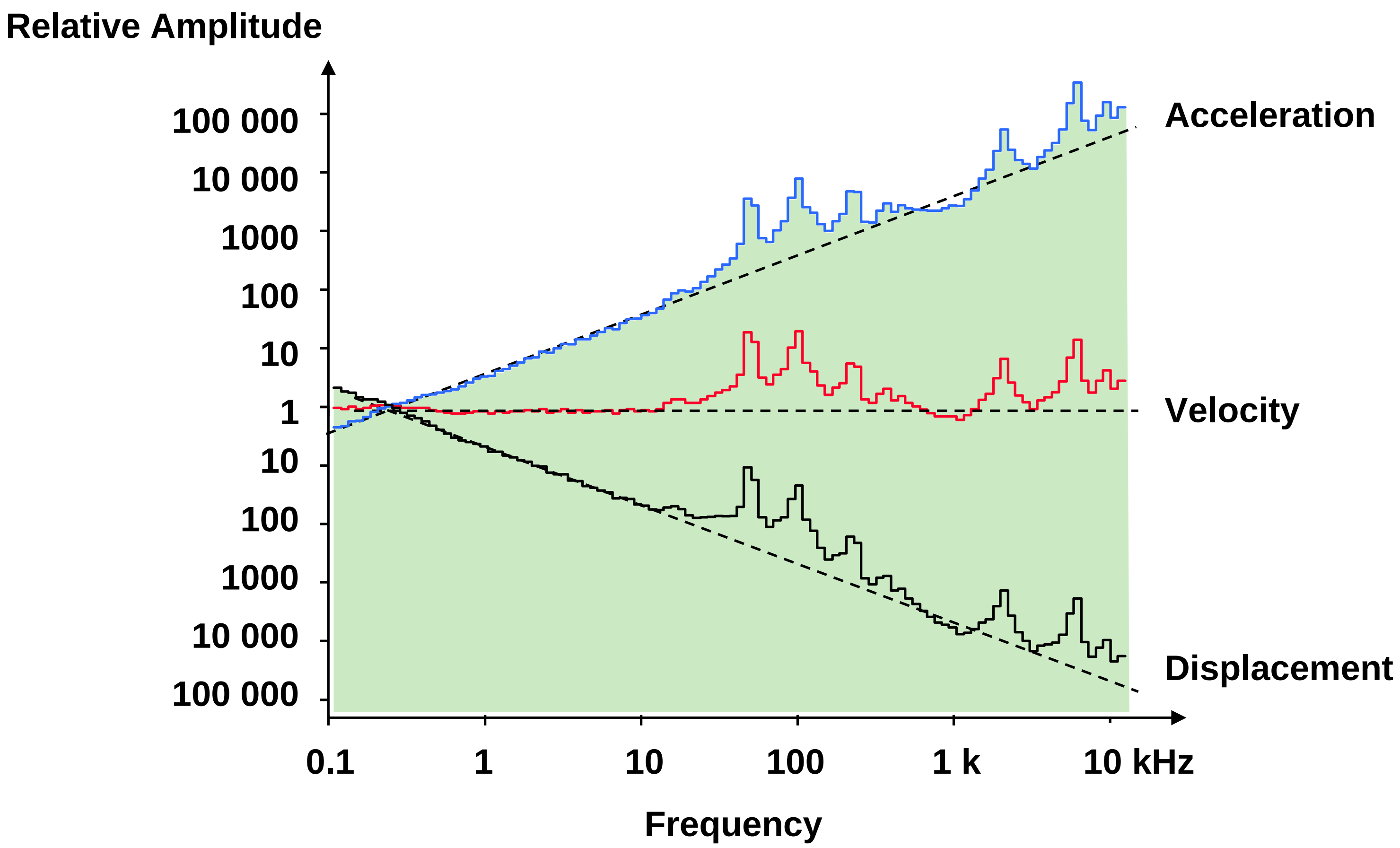
2.4 Vibration Measurement

- “Real World” Vibration Levels



2.4 Vibration Measurement

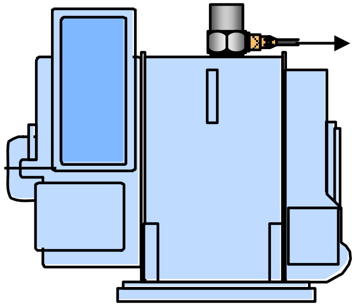
- Vibration Parameters



2.4 Vibration Measurement

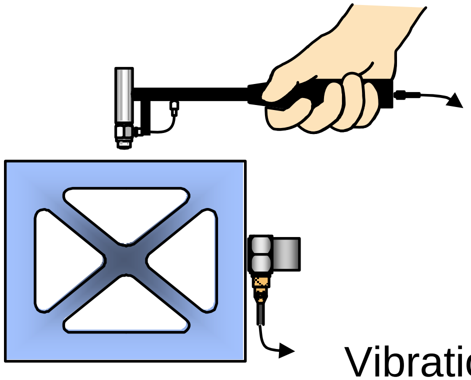
- Signal vs. System Analysis

Signal Analysis



Vibration signal

System Analysis



Excitation (Input)

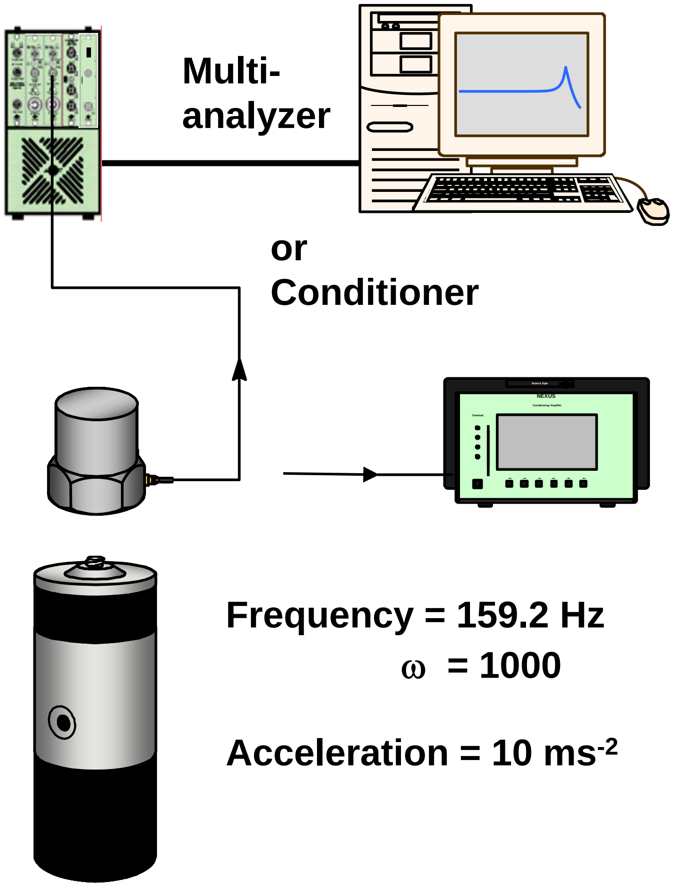
Vibration Response (Output)

2.4 Vibration Measurement

- Calibrator usage

Check System Set-up

- Use a calibrator to provide a well known input to a system, e.g. a Multi-Analyzer or conditioner and read out.
- Check that the read out corresponds to the input
- Change settings to give precise results



Multi-analyzer

or Conditioner

Frequency = 159.2 Hz

ω = 1000

Acceleration = 10 ms^{-2}

2.5 Vibration Isolation

- Machine mounting for vibration attenuation

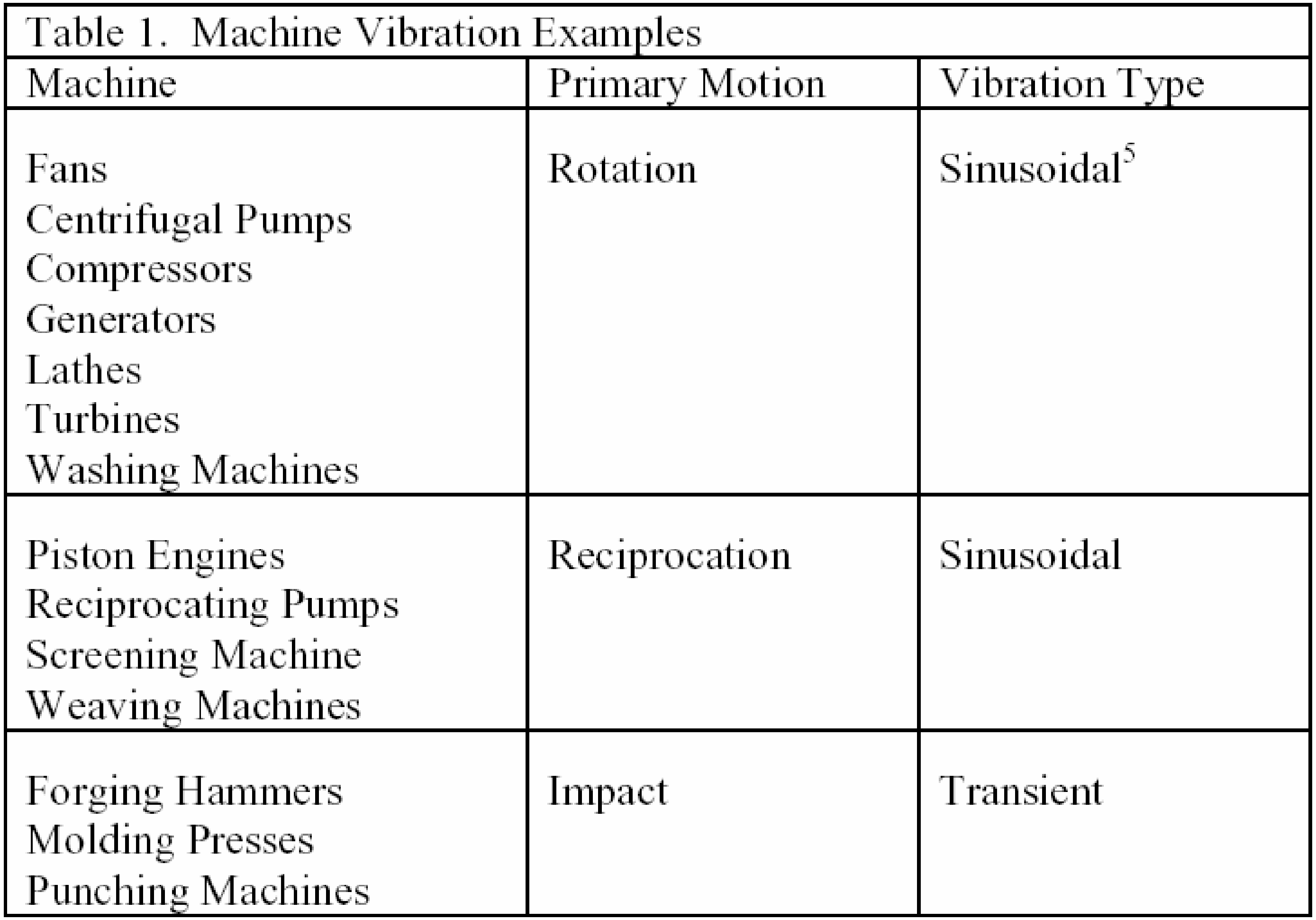


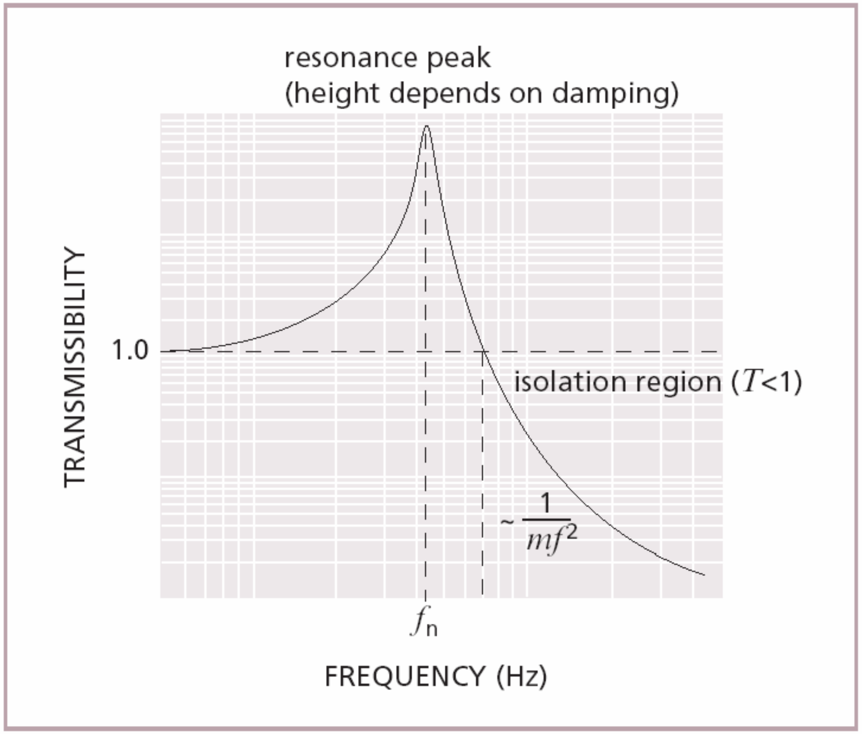
Table 1. Machine Vibration Examples

Machine	Primary Motion	Vibration Type
Fans Centrifugal Pumps Compressors Generators Lathes Turbines Washing Machines	Rotation	Sinusoidal[5]
Piston Engines Reciprocating Pumps Screening Machine Weaving Machines	Reciprocation	Sinusoidal
Forging Hammers Molding Presses Punching Machines	Impact	Transient

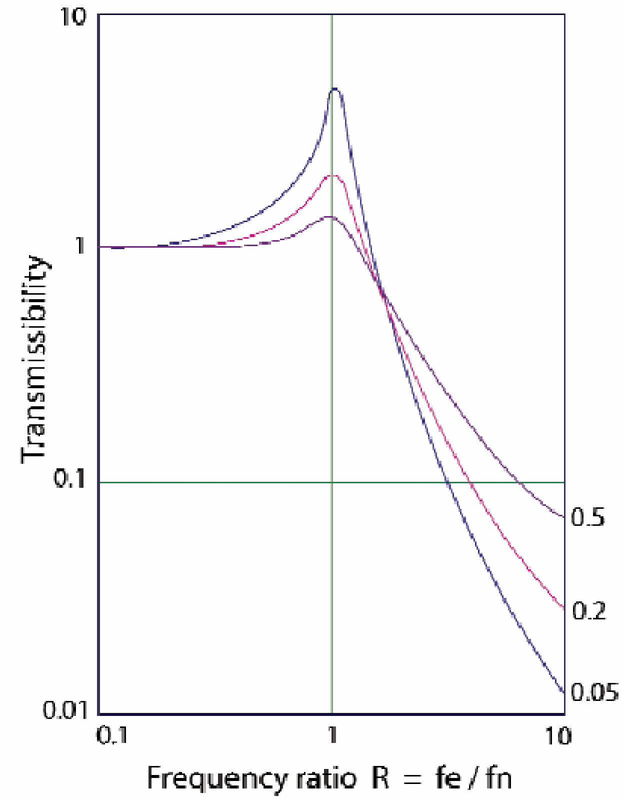
2.5 Vibration Isolation

- Transmissibility

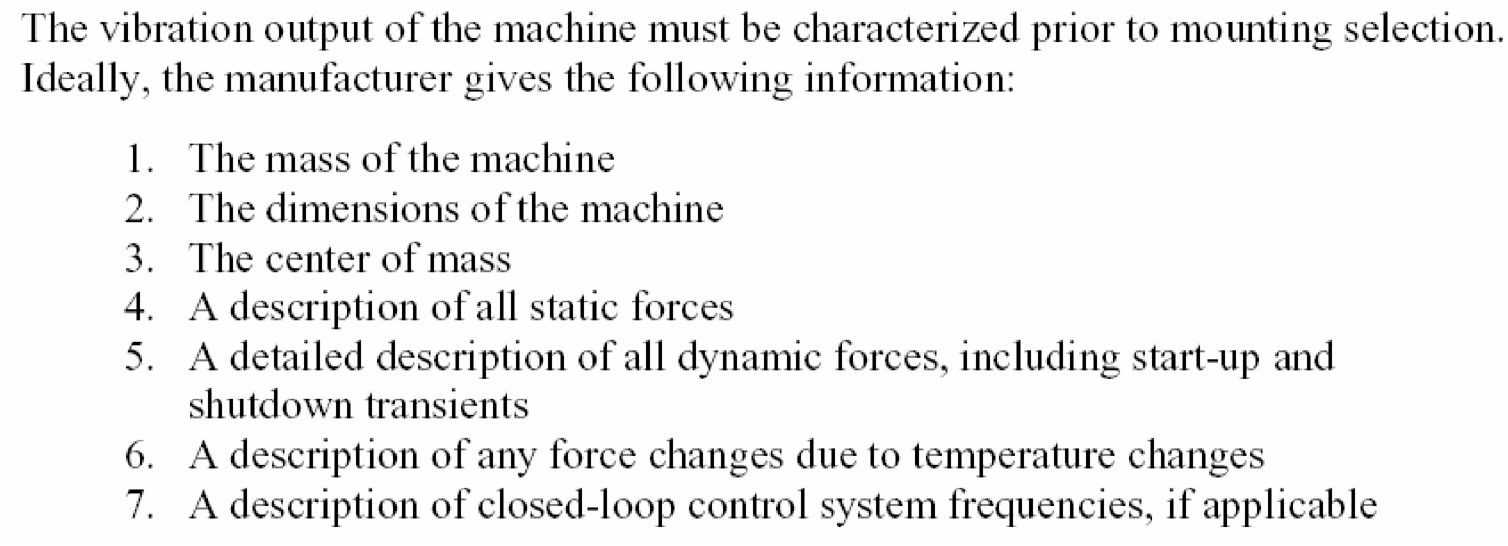
$$T = \sqrt{\frac{1 + \left(2\zeta f / f_n\right)^2}{\left(1 - f^2 / f_n^2\right)^2 + \left(2\zeta f / f_n\right)^2}}$$



A typical transmissibility vs frequency curve for a system with one degree of freedom

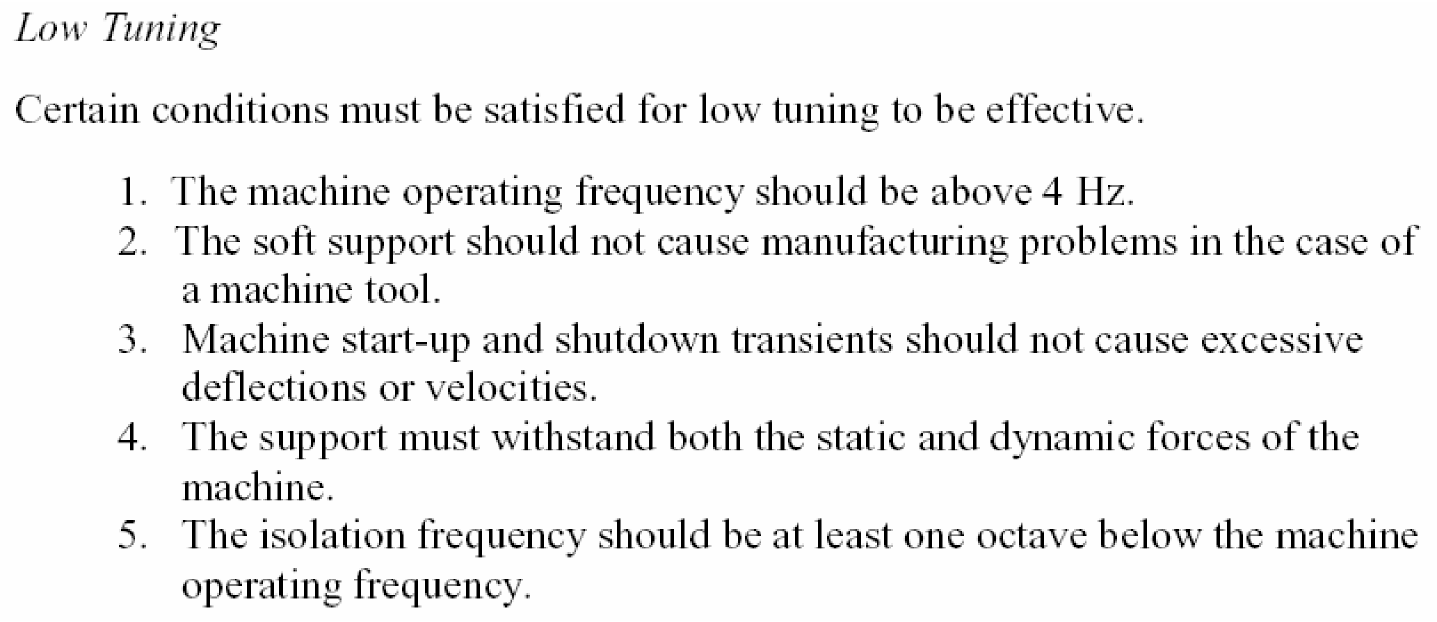


2.5 Vibration Isolation



The vibration output of the machine must be characterized prior to mounting selection. Ideally, the manufacturer gives the following information:

1. The mass of the machine
2. The dimensions of the machine
3. The center of mass
4. A description of all static forces
5. A detailed description of all dynamic forces, including start-up and shutdown transients
6. A description of any force changes due to temperature changes
7. A description of closed-loop control system frequencies, if applicable



Low Tuning

Certain conditions must be satisfied for low tuning to be effective.

1. The machine operating frequency should be above 4 Hz.
2. The soft support should not cause manufacturing problems in the case of a machine tool.
3. Machine start-up and shutdown transients should not cause excessive deflections or velocities.
4. The support must withstand both the static and dynamic forces of the machine.
5. The isolation frequency should be at least one octave below the machine operating frequency.

2.5 Vibration Isolation

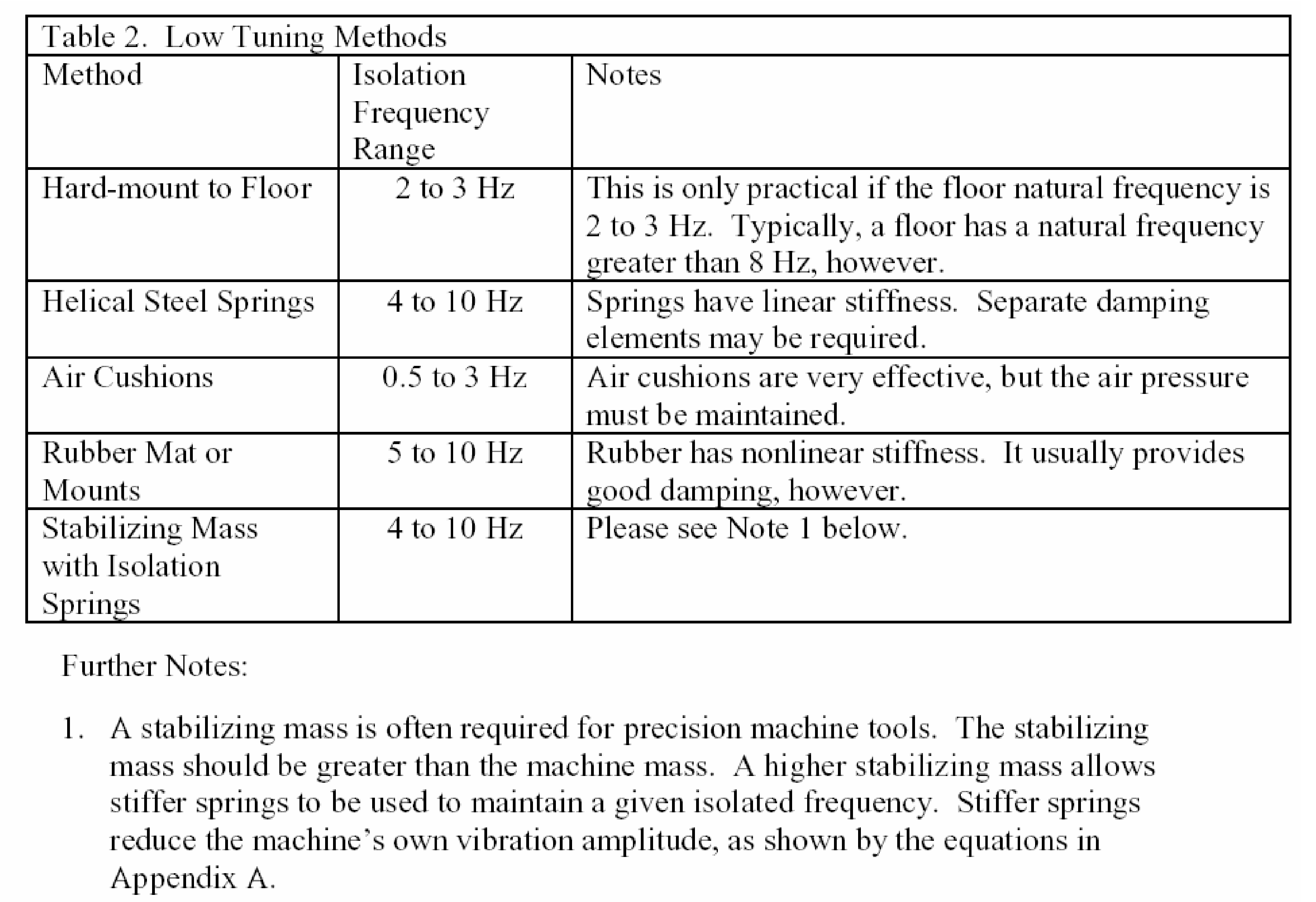


Table 2. Low Tuning Methods		
Method	Isolation Frequency Range	Notes
Hard-mount to Floor	2 to 3 Hz	This is only practical if the floor natural frequency is 2 to 3 Hz. Typically, a floor has a natural frequency greater than 8 Hz, however.
Helical Steel Springs	4 to 10 Hz	Springs have linear stiffness. Separate damping elements may be required.
Air Cushions	0.5 to 3 Hz	Air cushions are very effective, but the air pressure must be maintained.
Rubber Mat or Mounts	5 to 10 Hz	Rubber has nonlinear stiffness. It usually provides good damping, however.
Stabilizing Mass with Isolation Springs	4 to 10 Hz	Please see Note 1 below.

Further Notes:

1. A stabilizing mass is often required for precision machine tools. The stabilizing mass should be greater than the machine mass. A higher stabilizing mass allows stiffer springs to be used to maintain a given isolated frequency. Stiffer springs reduce the machine's own vibration amplitude, as shown by the equations in Appendix A.

2.5 Vibration Isolation

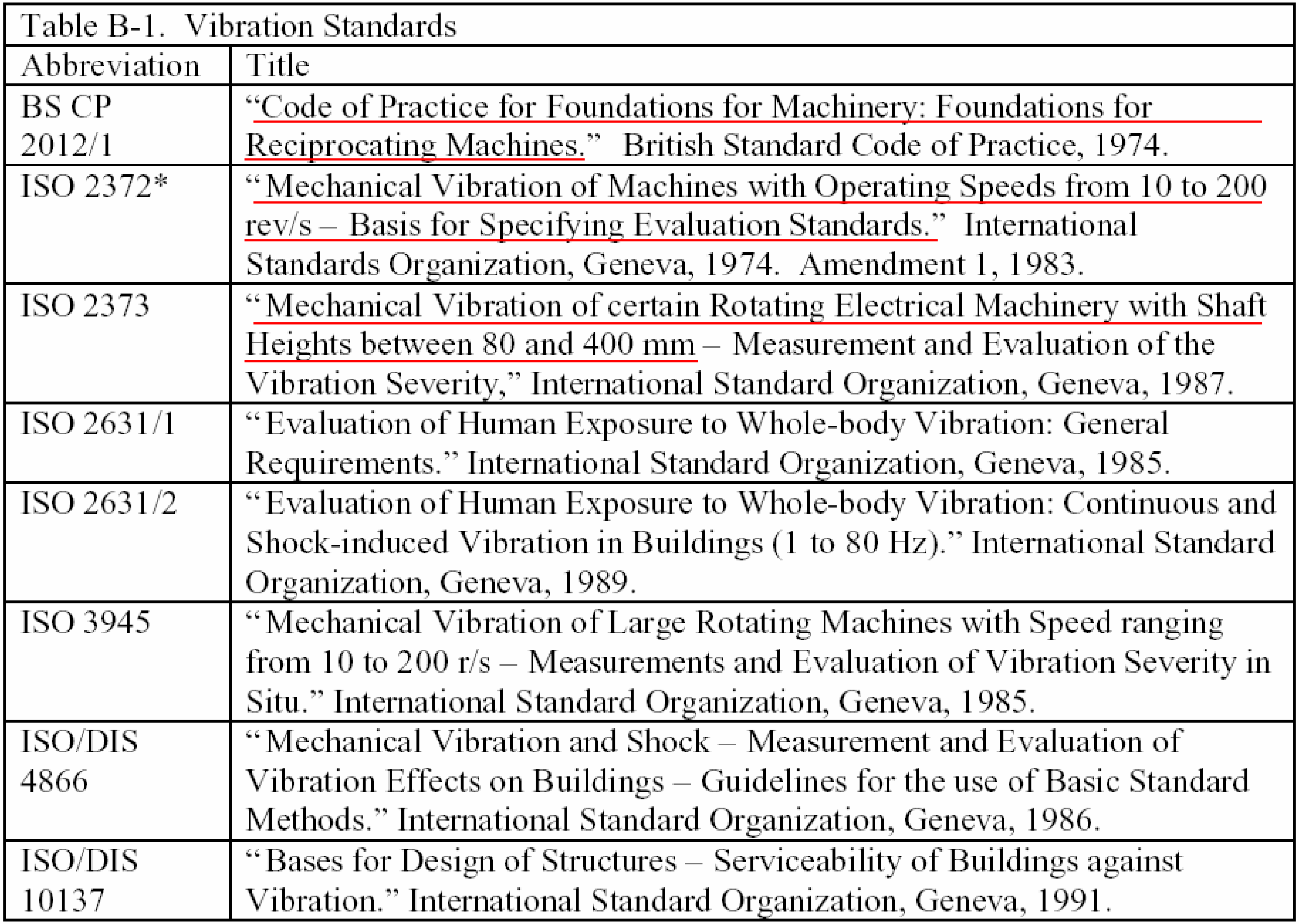


Table B-1. Vibration Standards

Abbreviation	Title
BS CP 2012/1	"Code of Practice for Foundations for Machinery: Foundations for Reciprocating Machines." British Standard Code of Practice, 1974.
ISO 2372*	"Mechanical Vibration of Machines with Operating Speeds from 10 to 200 rev/s – Basis for Specifying Evaluation Standards." International Standards Organization, Geneva, 1974. Amendment 1, 1983.
ISO 2373	"Mechanical Vibration of certain Rotating Electrical Machinery with Shaft Heights between 80 and 400 mm – Measurement and Evaluation of the Vibration Severity," International Standard Organization, Geneva, 1987.
ISO 2631/1	"Evaluation of Human Exposure to Whole-body Vibration: General Requirements." International Standard Organization, Geneva, 1985.
ISO 2631/2	"Evaluation of Human Exposure to Whole-body Vibration: Continuous and Shock-induced Vibration in Buildings (1 to 80 Hz)." International Standard Organization, Geneva, 1989.
ISO 3945	"Mechanical Vibration of Large Rotating Machines with Speed ranging from 10 to 200 r/s – Measurements and Evaluation of Vibration Severity in Situ." International Standard Organization, Geneva, 1985.
ISO/DIS 4866	"Mechanical Vibration and Shock – Measurement and Evaluation of Vibration Effects on Buildings – Guidelines for the use of Basic Standard Methods." International Standard Organization, Geneva, 1986.
ISO/DIS 10137	"Bases for Design of Structures – Serviceability of Buildings against Vibration." International Standard Organization, Geneva, 1991.

2.5 Vibration Isolation

- Suspension system of vehicle

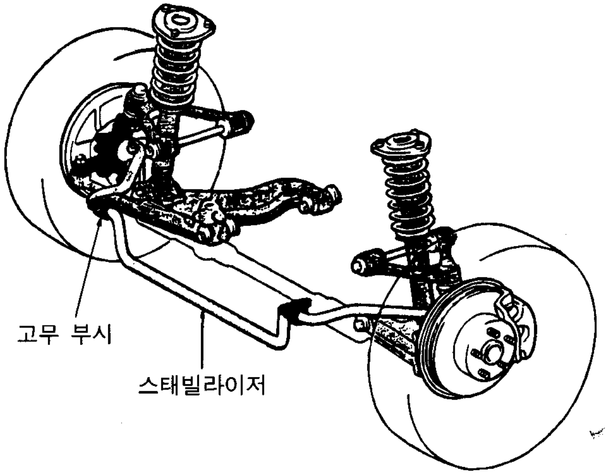


그림 3-31 스태빌라이저

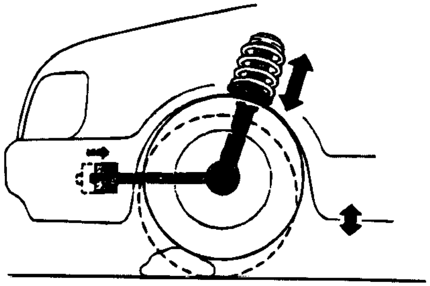
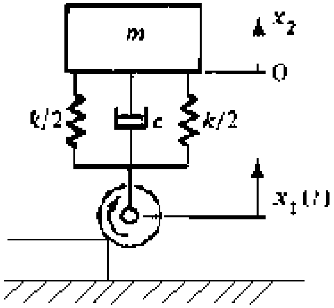
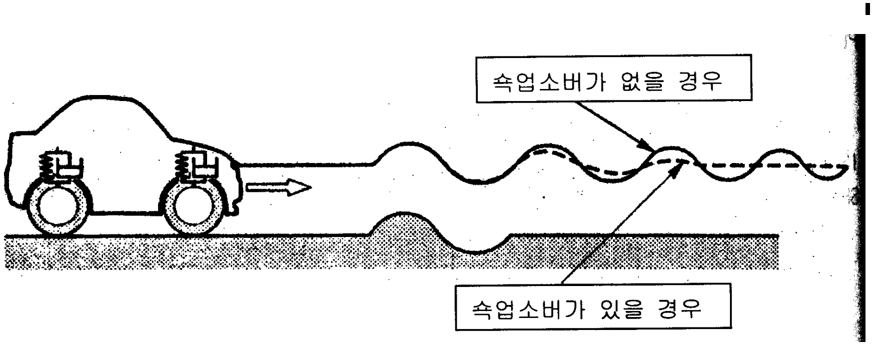


그림 3-3 충격과 진동의 완화





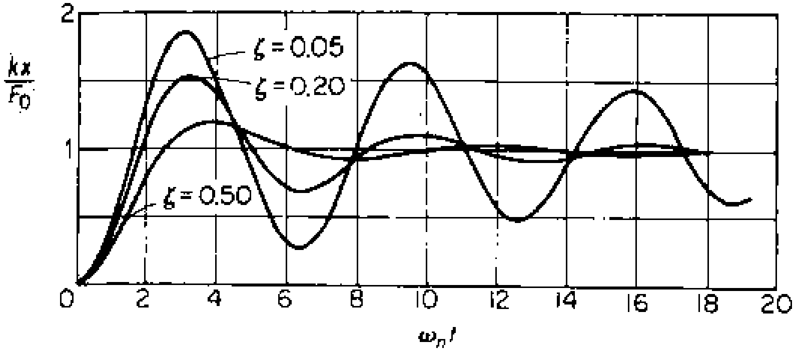
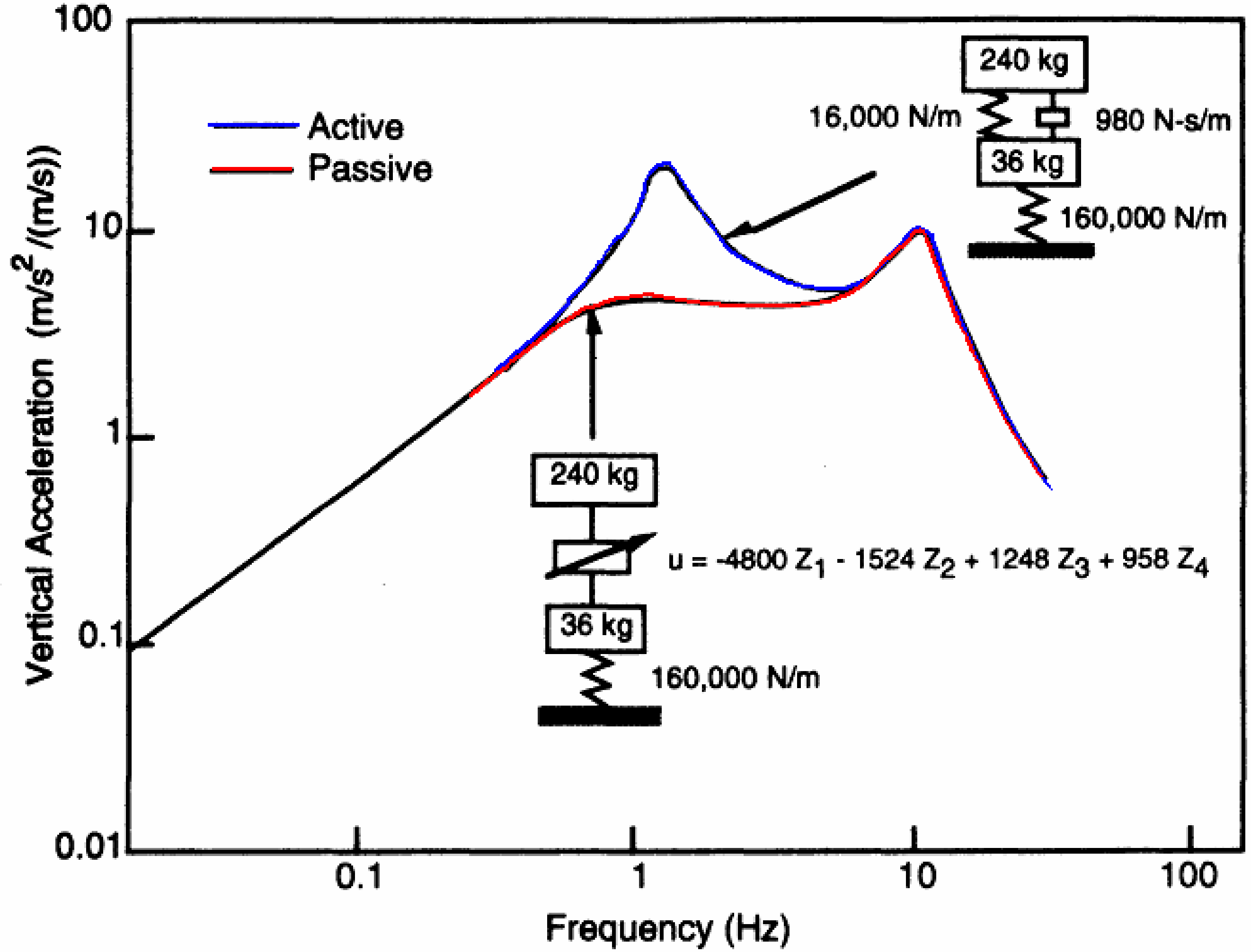


Figure 4.2-3. Response to a unit step function.

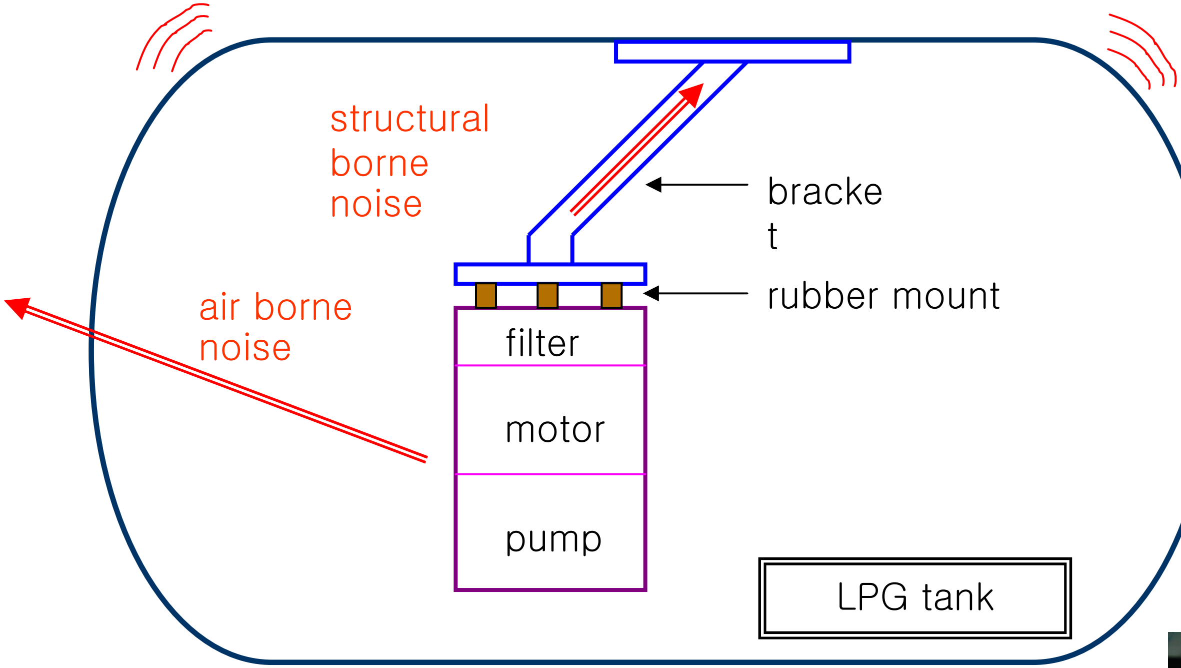
2.5 Vibration Isolation

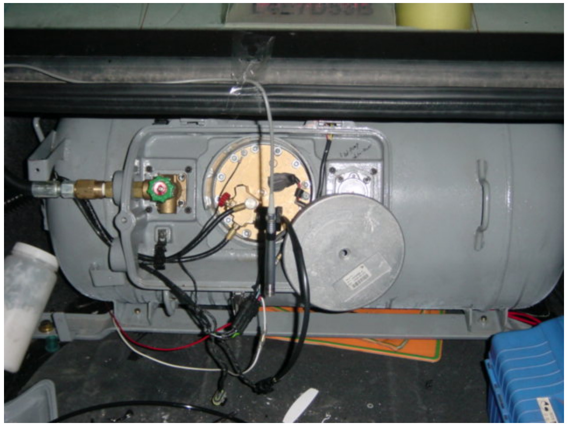
- Suspension system of vehicle



2.5 Vibration Isolation

- LPI System (Liquid propane injection system)

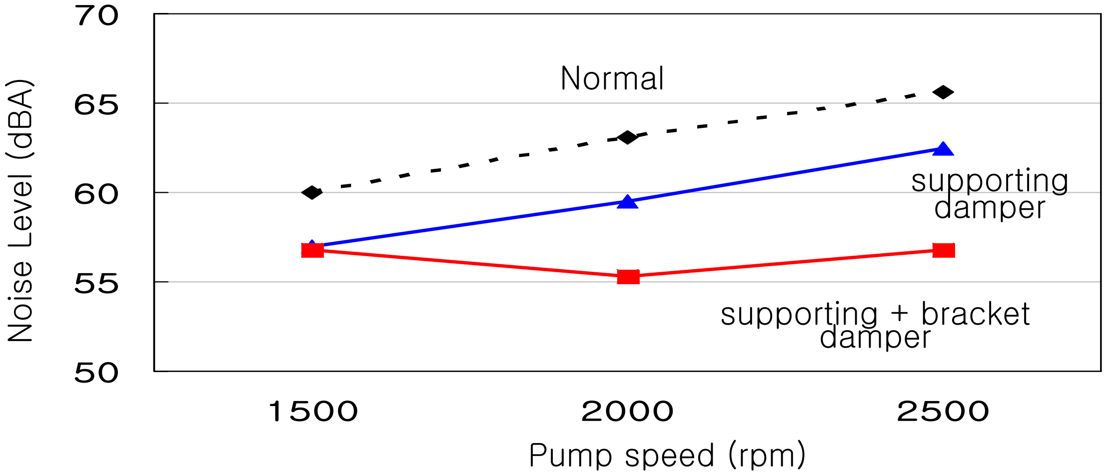




2.5 Vibration Isolation

Type rpm	Normal	supporting damper	supporting + bracket damper
1500	60.0	57.0	56.8
2000	63.1	59.5	55.3
2500	65.6	62.5	56.8

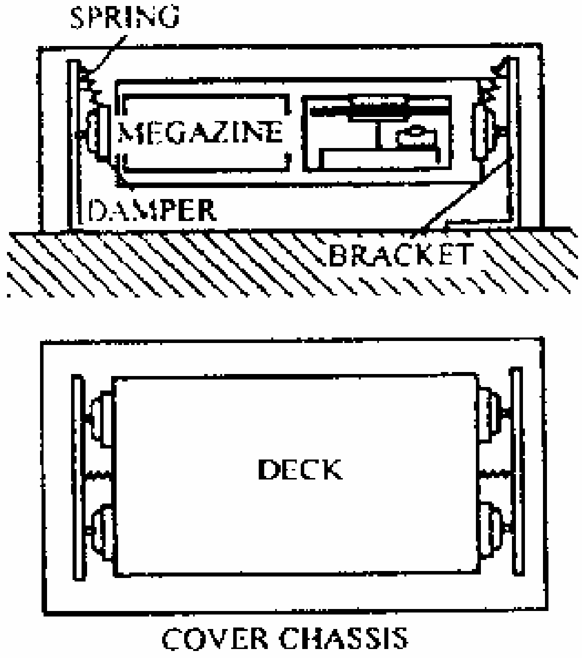
Noise Level (dBA)



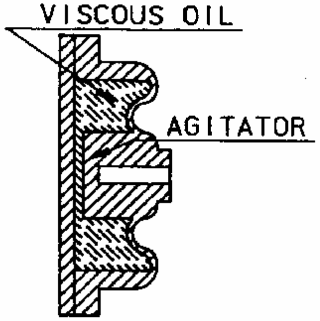
- ▷ supporting damper : −3.1 dB (2500rpm)
- ▷ supporting + bracket damper: –8.8 dB (2500rpm)

2.5 Vibration Isolation

- Suspension design of car CD player



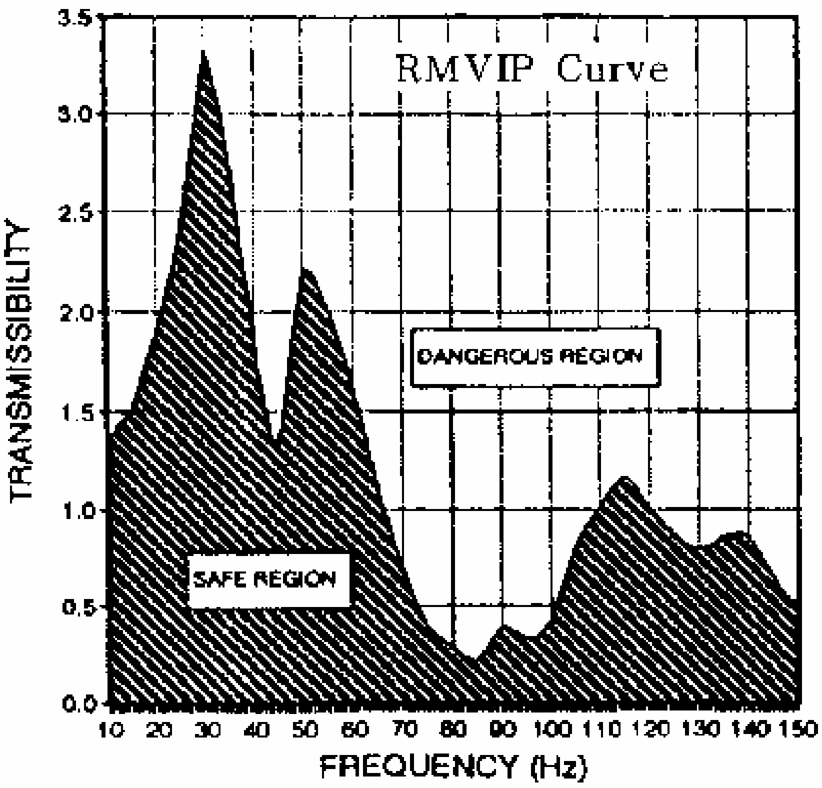
Suspension structure of car CD changer



Structure of oil dampers

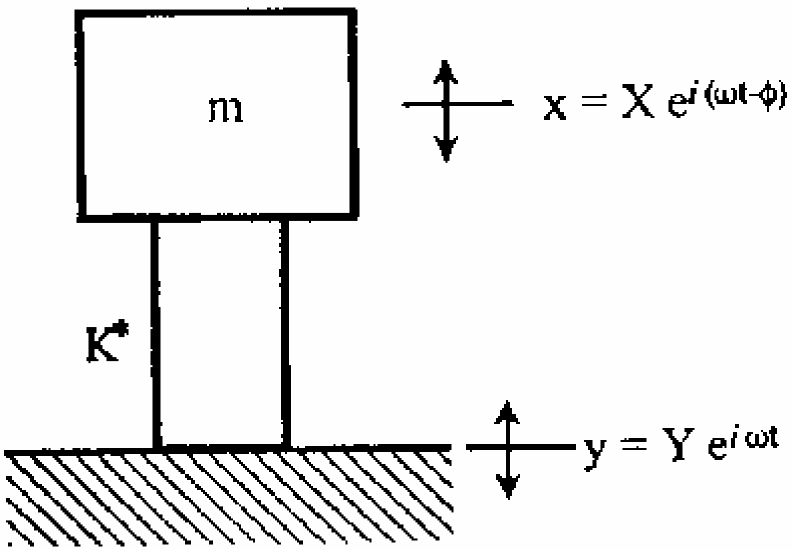
Required Minimum Vibration Isolation Performance Curve, RMVIP CURVE

$$RMVIP\ CURVE = \frac{NON-ISOLATED\ VIBRATION\ ENDURANCE\ LIMIT}{VIBRATION\ LEVEL\ OF\ ENVIRONMENT}$$



2.5 Vibration Isolation

- 1 DOF model with a complex stiffness



Vibration isolation system with mass and complex stiffness

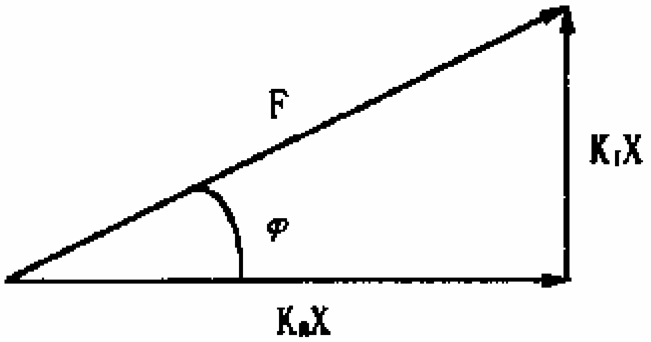
$$m\ddot{x} = -K^*(x - y)$$

$$K^* = K_R + K_I i$$

$$x = X e^{i(\omega t - \phi)}$$

$$y = Y e^{i\omega t}$$

$$\left\|\frac{X}{Y}\right\| = \sqrt{\frac{K_R^2 + K_I^2}{(K_R - M\omega)^2 + K_I^2}}$$

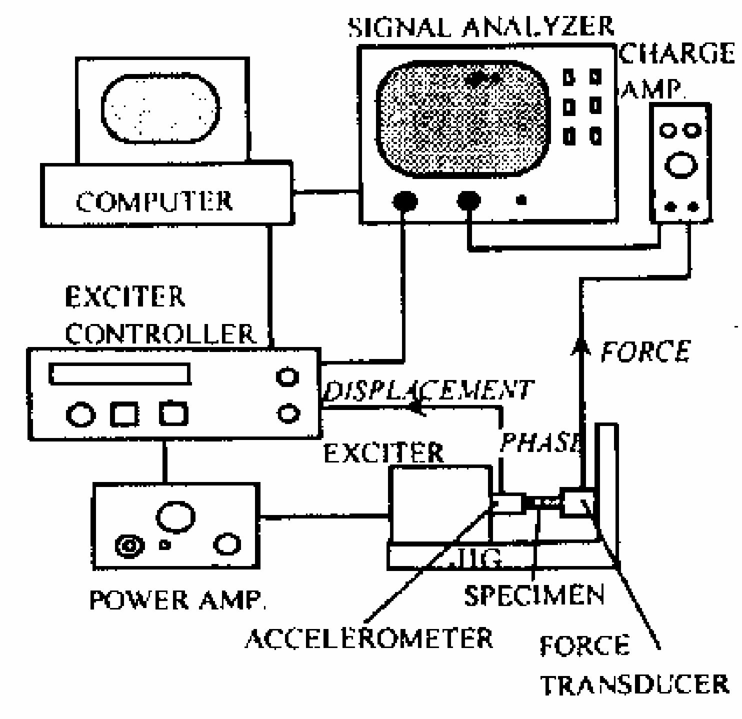


$$K_R = \frac{F}{X}\cos\phi$$

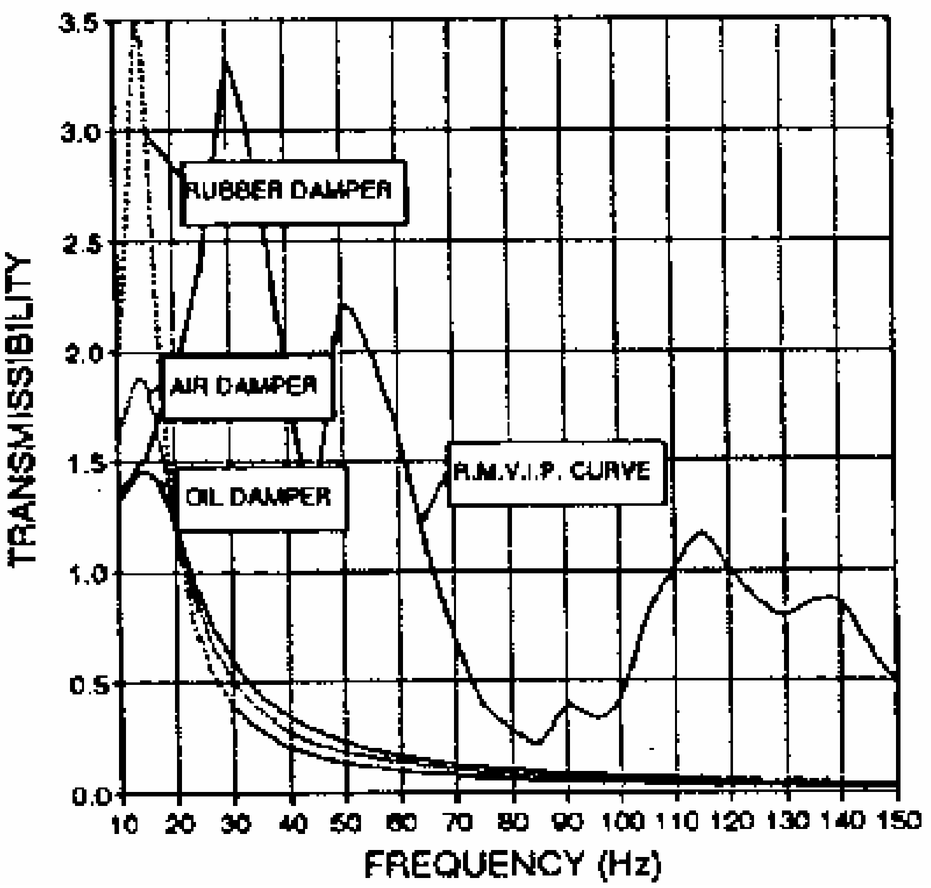
$$K_I = \frac{F}{X}\sin\phi$$

2.5 Vibration Isolation

- Complex stiffness measurement and transmissibility

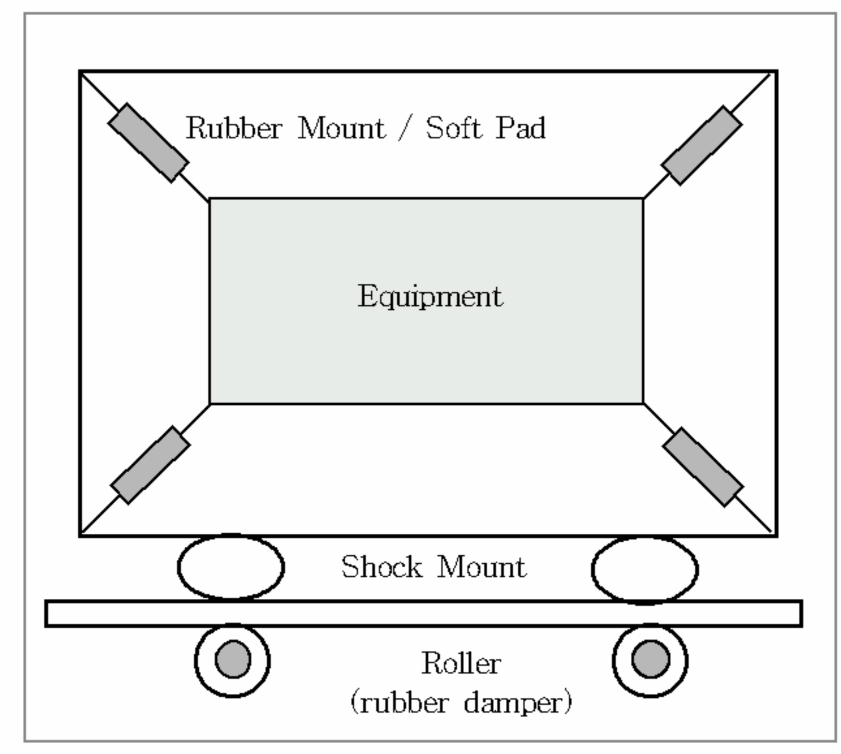


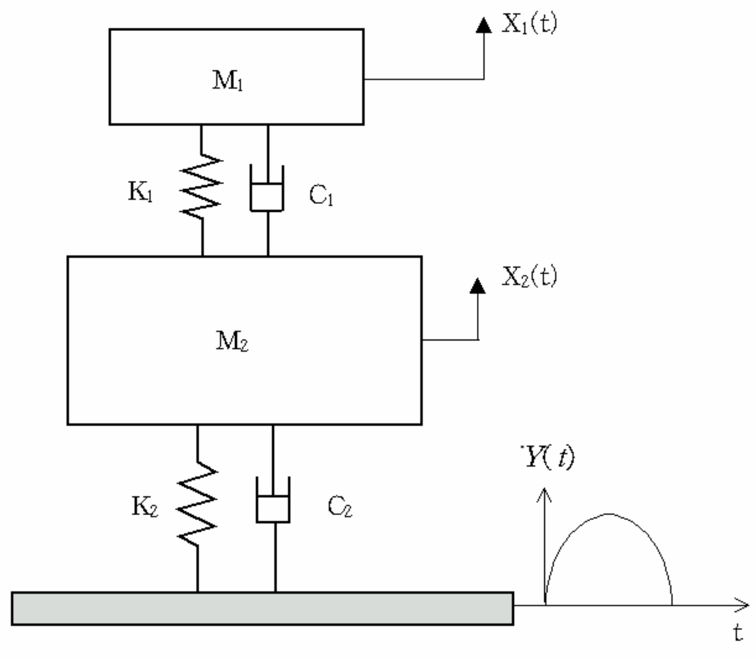
Set-up for complex stiffness measurement and transmissibility calculation



2.5 Vibration Isolation

- Shock mount design

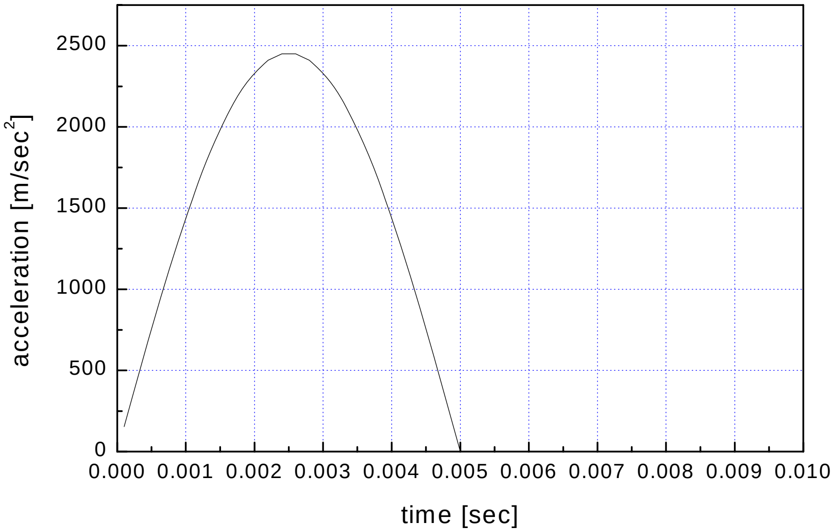


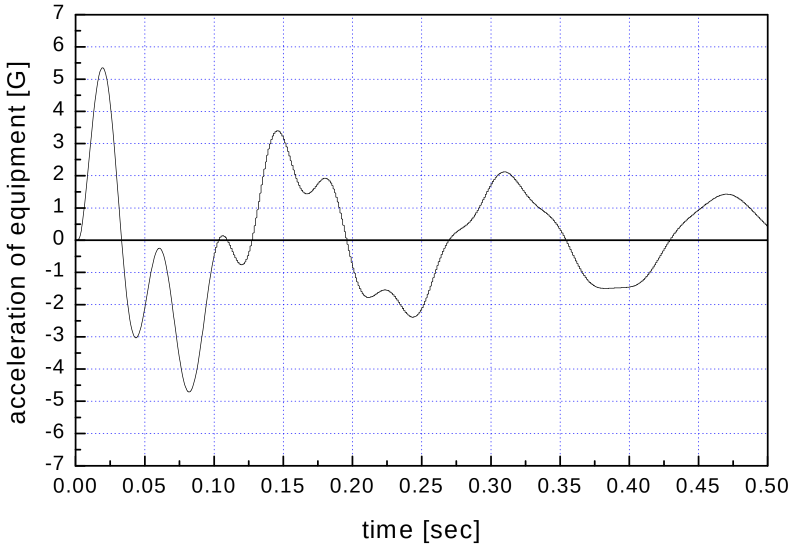


$$M_1\ddot{X}_1 + C_1(X_1 - X_2) + K_1(X_1 - X_2) = 0$$

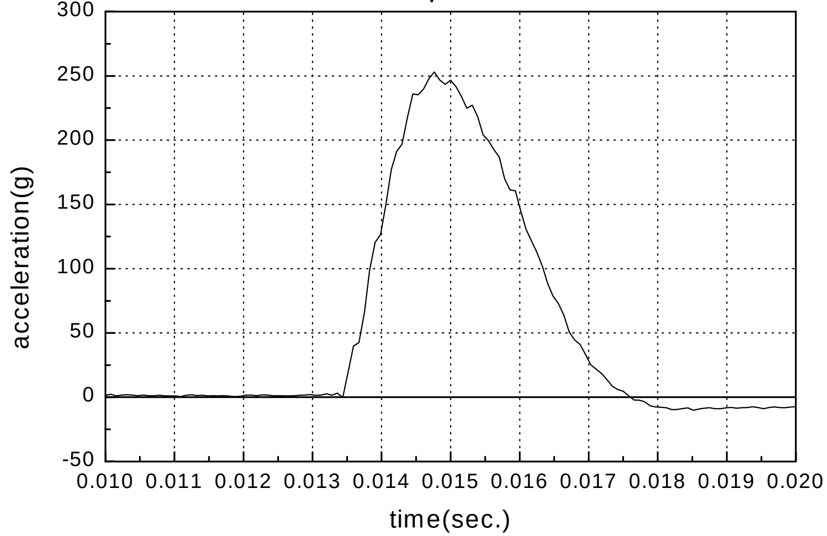
$$M_2\ddot{X}_2 + C_1(X_2 - X_1) + K_1(X_2 - X_1) + C_2(X_2 - Y) + K_2(X_2 - Y) = 0$$

2.5 Vibration Isolation

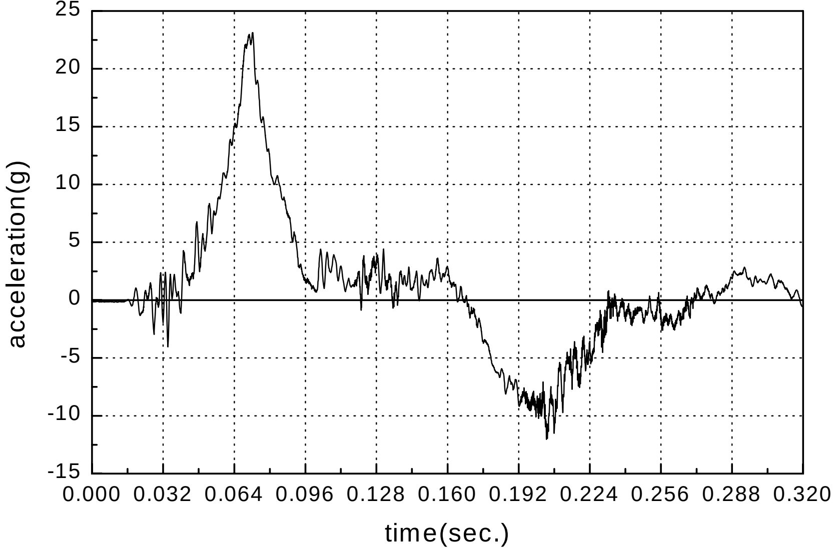




input

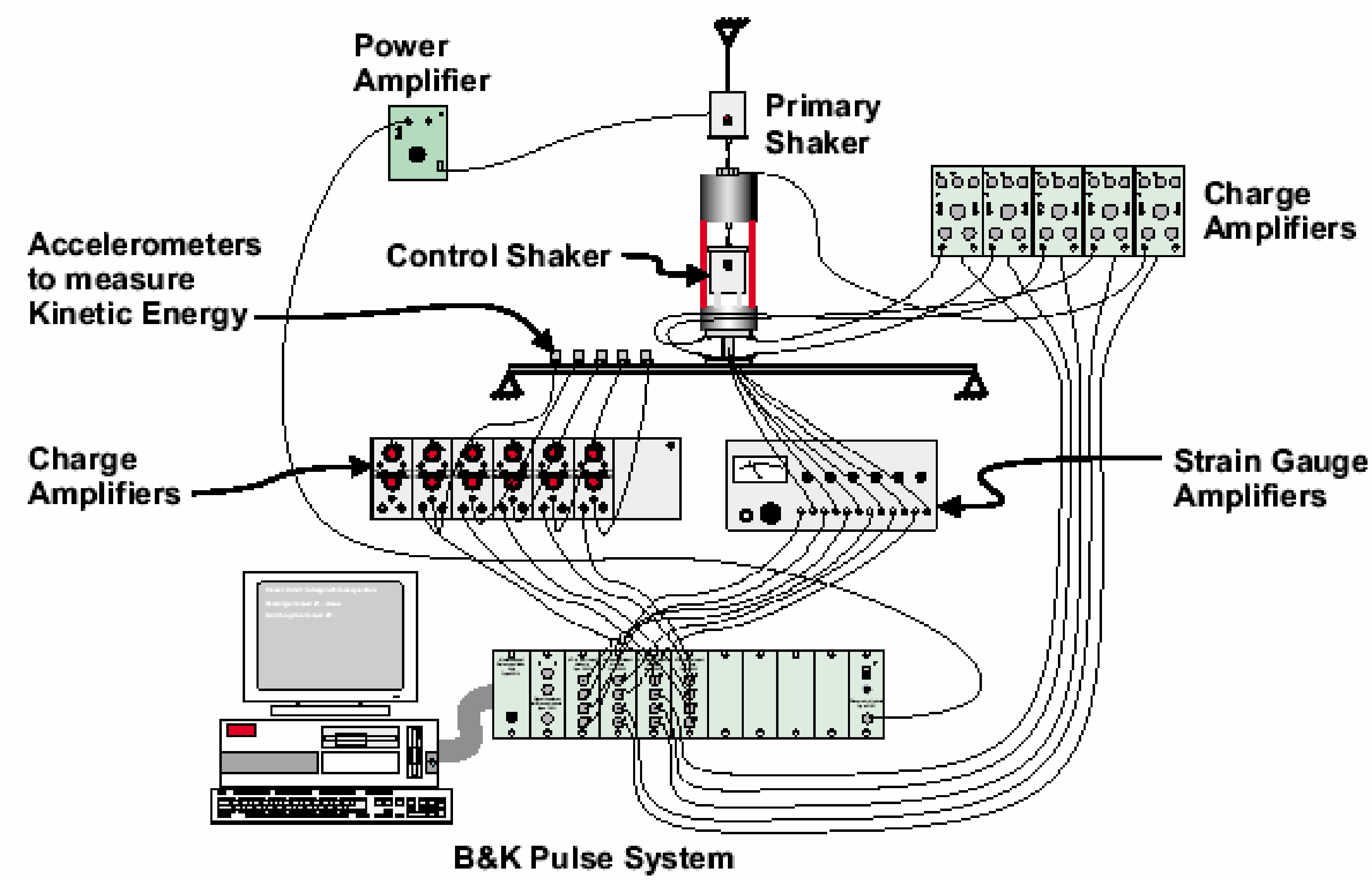


response



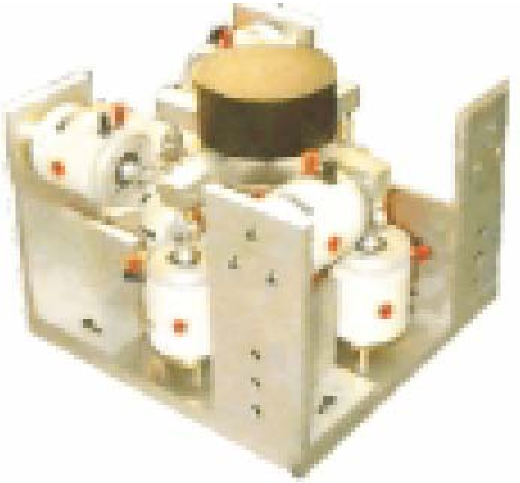
2.5 Vibration Isolation

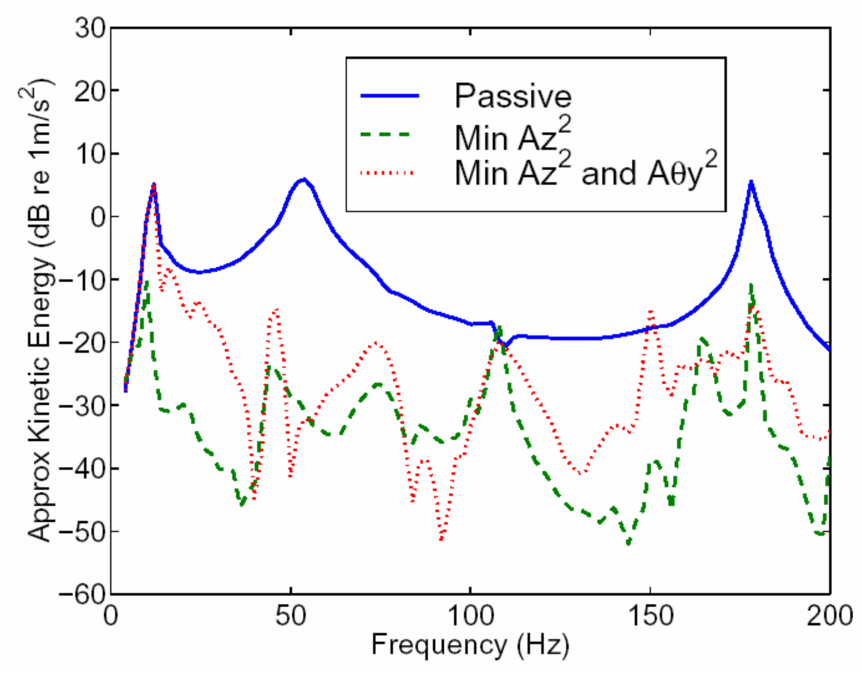
- Active vibration control of beam

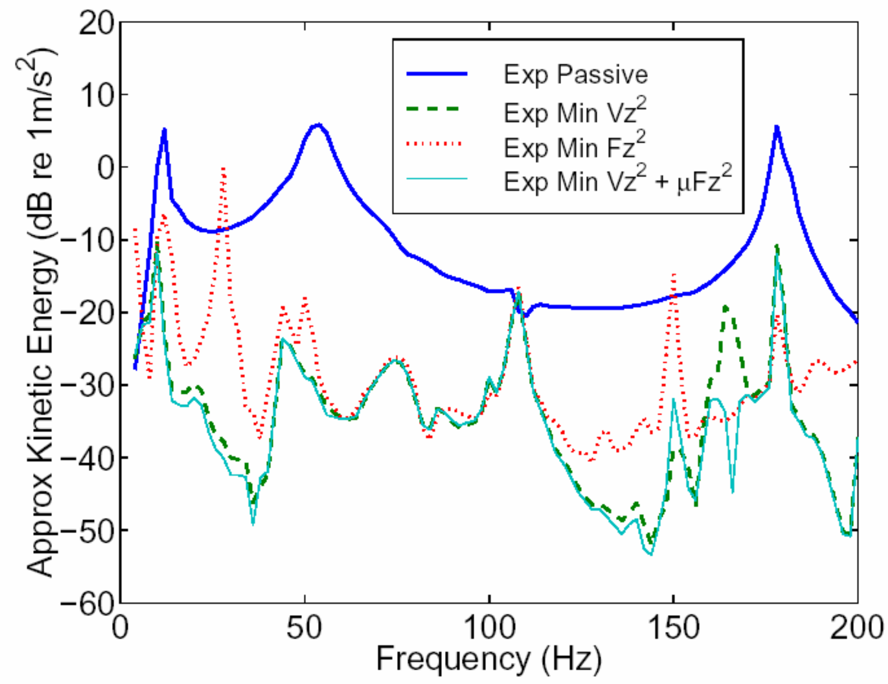


2.5 Vibration Isolation

- Active vibration control

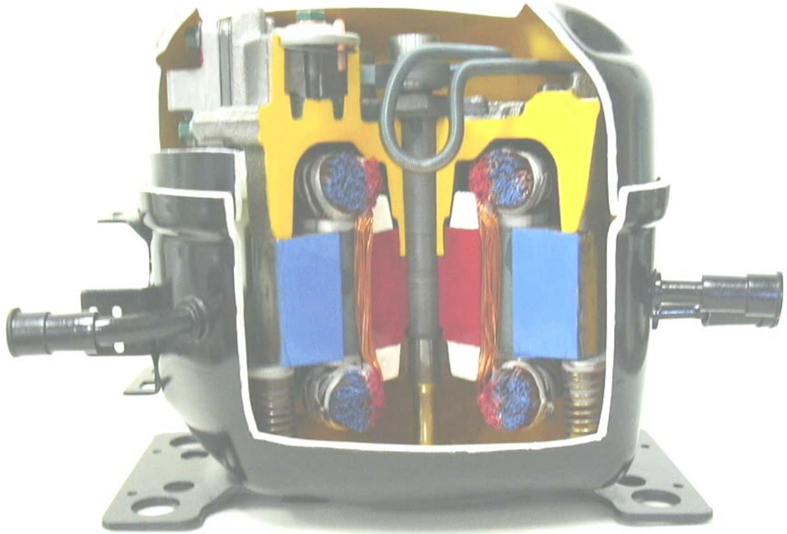


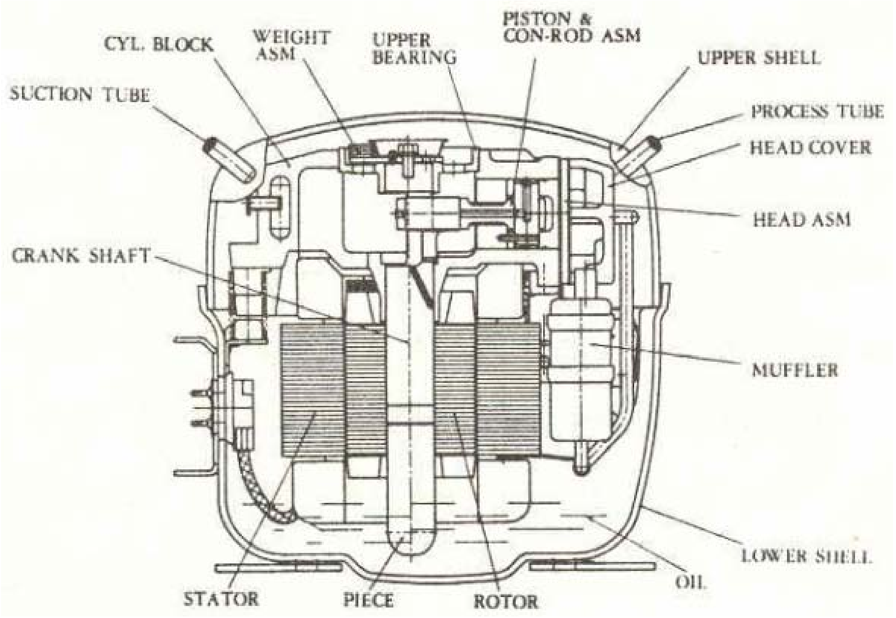




2.5 Vibration Isolation

- Reciprocating comp.

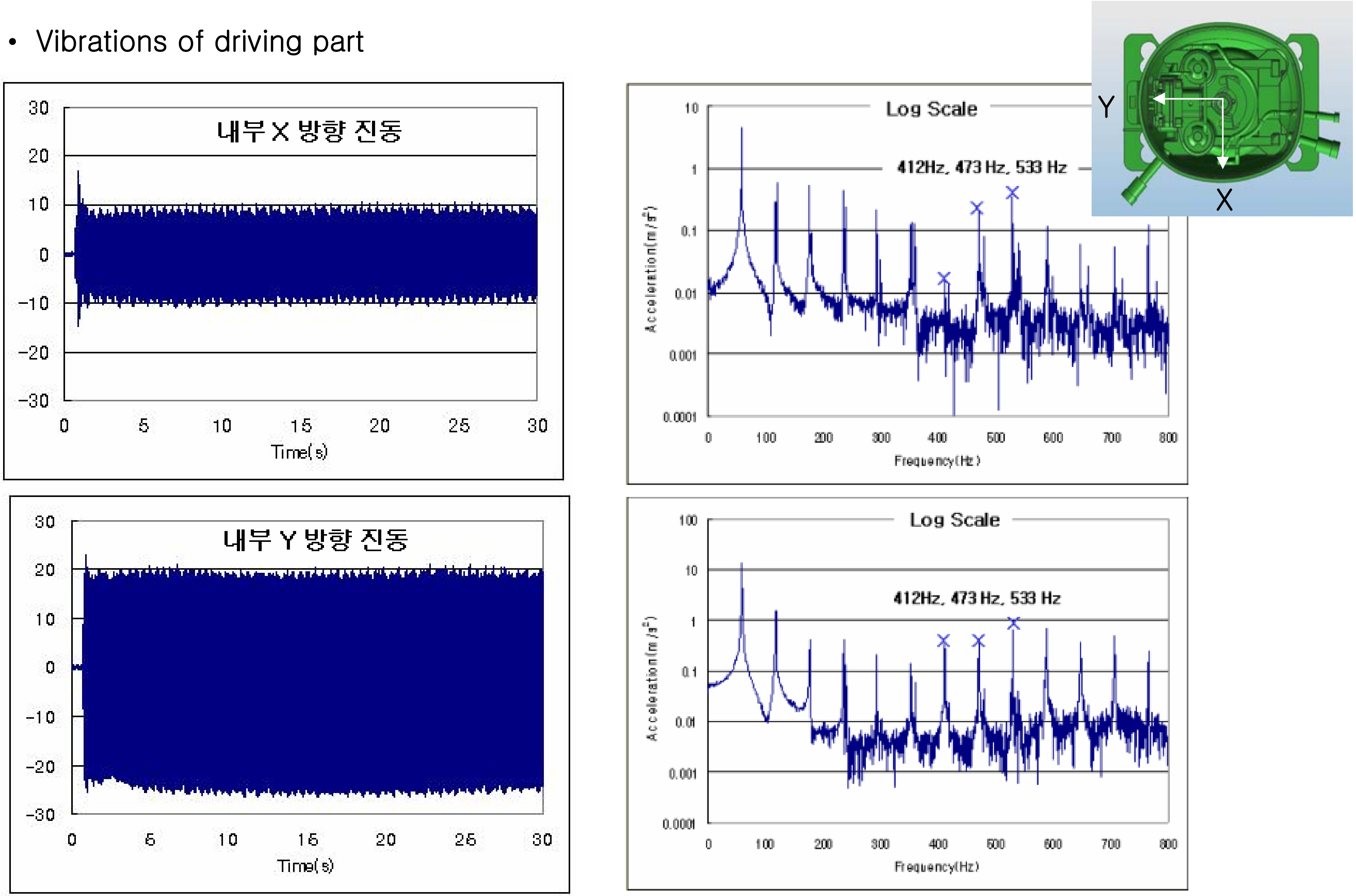




  - 6 DOF rigid model
  - Reciprocating force ⇒ Fundamental frequency(ω_o), Super harmonics($2\omega_o$, $3\omega_o$, ,,,)
  - Horizontal direction excitation

2.5 Vibration Isolation

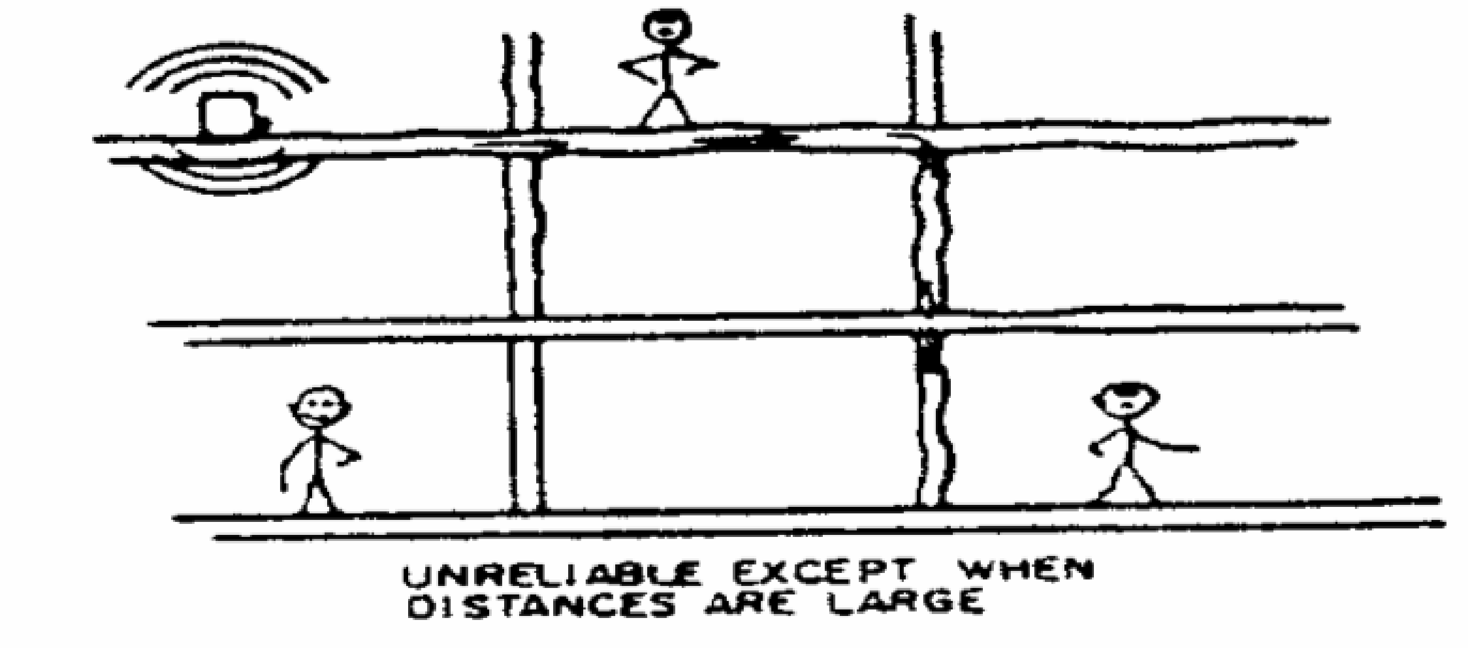
- Vibrations of driving part



2.5 Vibration Isolation

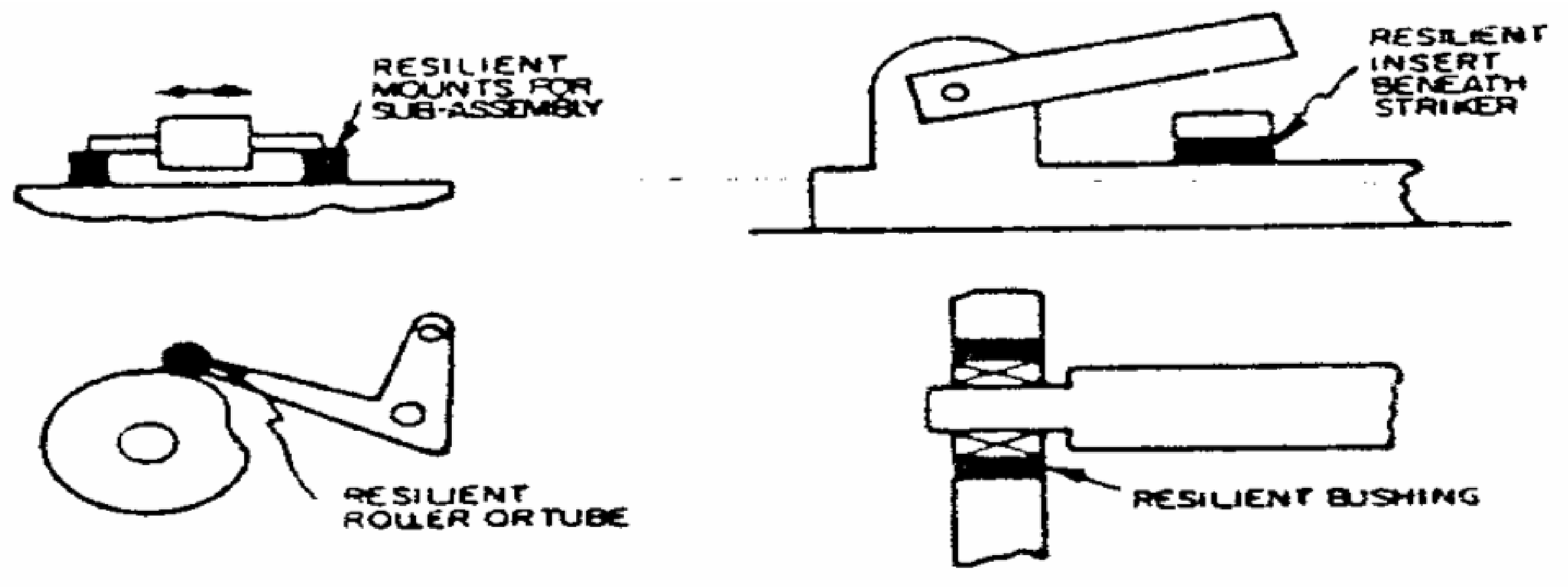
- Vibration isolation method

1. Separating source and receiver



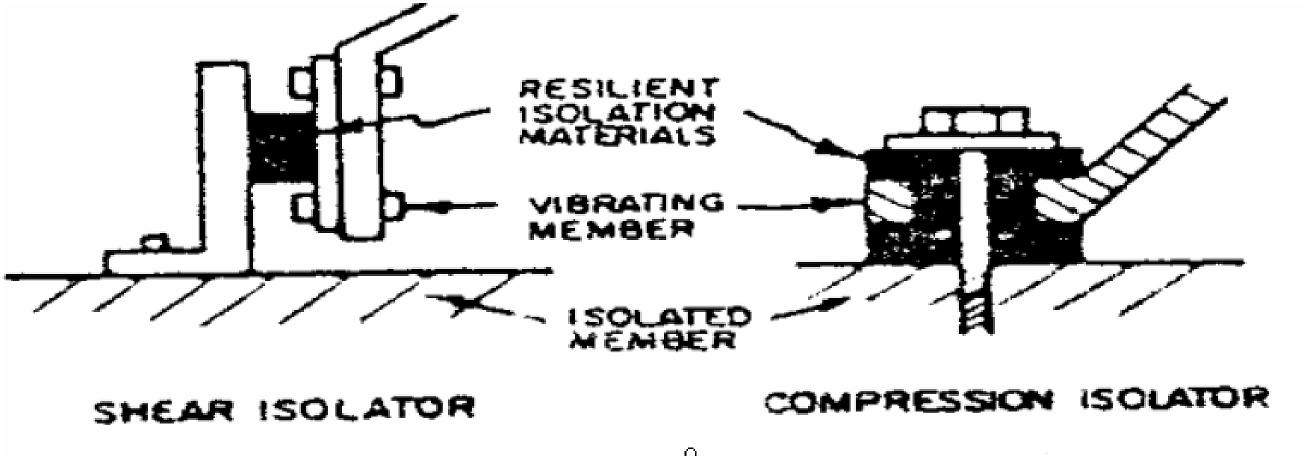
2.5 Vibration Isolation

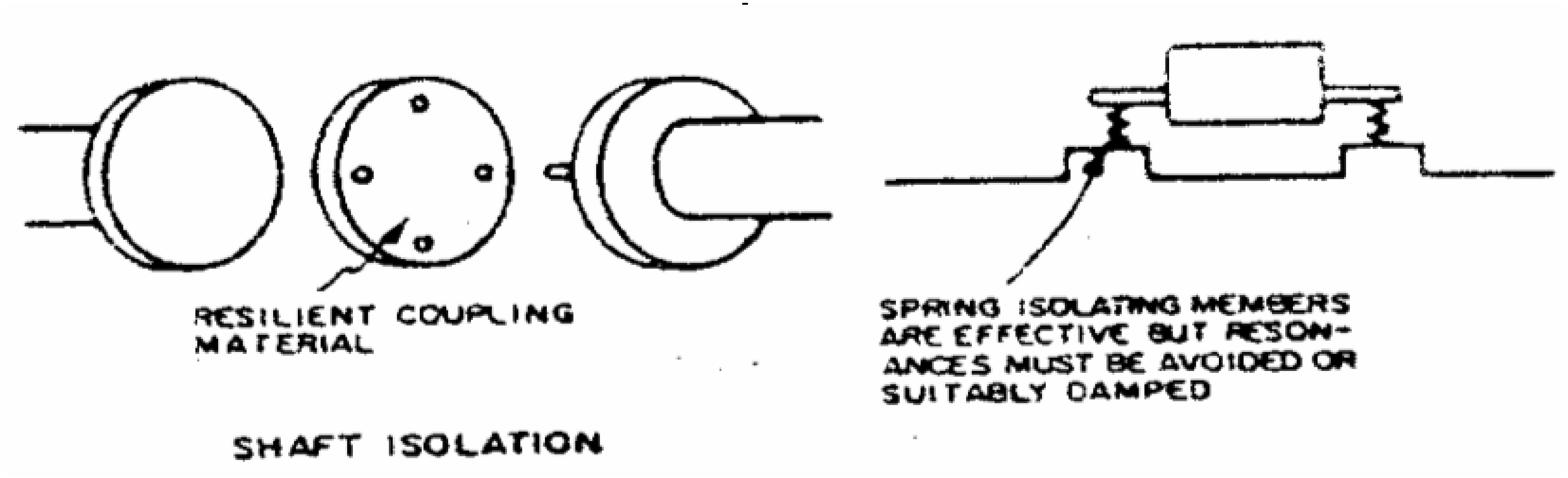
2. Vibration break (physical break in solid structure)



A VIBRATION BREAK MUST BE SOFT COMPARED TO THE VIBRATING PART

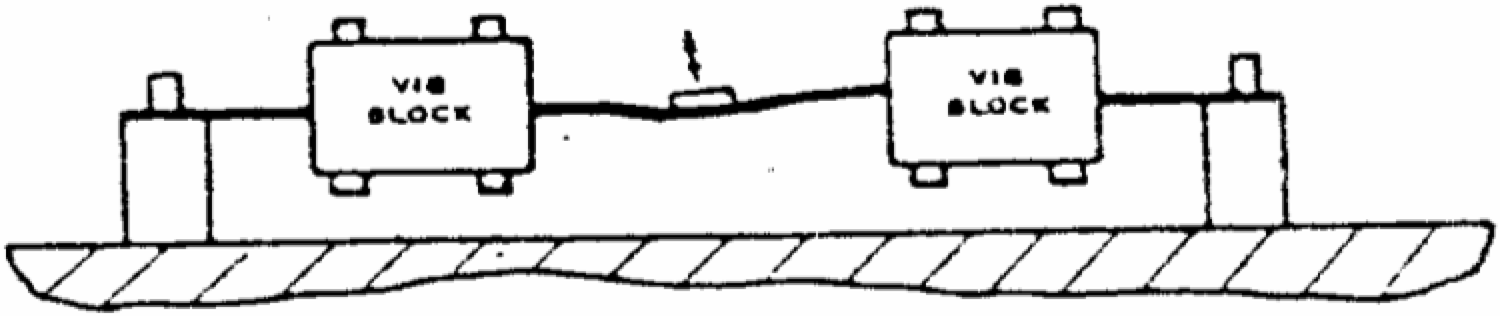
2.5 Vibration Isolation





2.5 Vibration Isolation

3. Vibration block



A VIBRATION BLOCK MUST BE VERY MASSIVE COMPARED TO THE VIBRATING SOURCE

4. Vibration damping